Borel Sets and Analytic Sets

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1 Introduction

In this essay, we will first develop the basic theory of Borel sets, then extend this to analytic sets. We will see some interesting properties, which will give us a deeper understanding of the sets in Polish Spaces.

The properties we introduce in this chapter can usually be established effectively without the Axiom of Choice. However, in descriptive set theory one frequently considers unions and intersections of countably many sets of reals, and we shall often use facts like 'the union of countably many countable sets is countable'. Thus we shall work throughout this chapter, under ZF+ Countable Axiom of Choice.

2 Definitions

Seq is the set of all finite sequences of natural numbers.

When talking about members of *Seq*, we will sometimes make use of the $s \frown k$ notation to mean appending k to s, where s is a member of *Seq*. So for example, if $s = 01$, $s \frown 1 = 011$.

For $s = \langle a_k : k < n \rangle \in \mathit{Seq}$ and $f = \langle c_k : k \in \mathbb{N} \rangle \in \mathcal{N}$, we say $s \subset f$ if: $(\forall k < n)\ c_k = a_k$.

Similarly, For $s = \langle a_k : k < n \rangle \in \mathit{Seq}$ and $f = \langle c_k : k < m \rangle \in \mathit{Seq}$, $m > n$, we say $s \subset f$ if: $(\forall k < n)\ c_k = a_k$

Baire Space is $\mathcal{N} = \omega^\omega$, i.e. all infinite sequence of natural numbers, with the following topology: For every finite sequence $s = \langle a_k : k < n \rangle$, let

$$O(s) = \{f \in \mathcal{N} : s \subset f\} = \{\langle c_k : k \in \mathbb{N} \rangle : (\forall k < n)\ c_k = a_k\}$$

$O(s)$ has an interesting property:

Claim. $O(s)$ is both open and closed.

Proof. We just need to show that $O(s)^c$ is open.

$$O(s)^c = \bigcup_{b \neq a} \{\langle c_k : k \in \mathbb{N}\rangle : (\forall k < n) c_k = b_k\} = \bigcup_{b \neq a} O(b)$$

Where $b = \langle b_k : k \in \mathbb{N}\rangle$. Thus $O(s)^c$ is open, so $O(s)$ is closed. $\square$

Remark. While *Seq* is countable, $\mathcal{N}$ is not, which we can prove via a diagonal argument. Indeed, $|\mathcal{N}| = 2^{\aleph_0}$.

An *algebra of sets* is a collection $\mathcal{S}$ of subsets of a given set S, such that:

(i) $S \in \mathcal{S}$

(ii) If $X \in \mathcal{S}$ and $Y \in \mathcal{S}$ then $X \cup Y \in \mathcal{S}$

(iii) If $X \in \mathcal{S}$, then $S - X \in \mathcal{S}$

A $\sigma-$*algebra* is an algebra, which is closed under countable unions (and as a result intersections).

Once we have the notion of an algebra ($\sigma-$algebra), we can also talk about an algebra ($\sigma-$algebra) which includes a collection $\mathcal{X}$ of subsets of S. We can then talk about the smallest algebra ($\sigma-$algebra) $\mathcal{S}$ such that $\mathcal{S} \supset \mathcal{X}$, namely the intersection of all algebra ($\sigma-$algebra) $\mathcal{S}$ such that $\mathcal{X} \subset \mathcal{S}$. It is easily checked that the countable intersection of algebras ($\sigma-$algebras) is still an algebra ($\sigma-$algebra).

3 Polish Space

A **Polish Space** is a topological space that is homeomorphic to a separable complete metric space. It has a lot of desirable properties: it is Hausdorff, for example.

Lemma 1. *Baire space is complete, separable and metrizable, thus a Polish space.*

Proof. For $f, g \in \mathcal{N}$, we let $f(n)$ and $g(n)$ be the nth element of f, g respectively. Consider the metric $d(f, g) = 1/2^{n+1}$, where n is the least number such that $f(n) \neq g(n)$. The countable set of all eventually constant sequences is dense in $\mathcal{N}$. This separable metric space is complete, as every Cauchy sequence converges. $\square$

Example 1 (Examples of Polish Space)**.** $\mathbb{R}$, $\mathcal{N}$, The unit interval $[0, 1]$, the unit circle T, Hilbert cube $[0, 1]^\omega$, etc

Now, the following lemma gives an important property of Polish space:

Lemma 2. *Let X be a Polish space, then there exist a continuous mapping from $\mathcal{N}$ onto X.*

Proof. Let X be a complete separable metric space; we construct a mapping f of $\mathcal{N}$ onto X as follows: By induction on the length of $s \in Seq$ we construct a collection $\{C_s : s \in Seq\}$ of closed balls such that $C_\emptyset = X$, and:

(i) $\text{diam}(C_s) \leq 1/n$ where $n = \text{length(s)}$

(ii) $C_s \subset \bigcup_{k=0}^{\infty} C_{s \frown k}$

(iii) If s $\subset t$, then center$(C_t) \in C_s$

In this construction, we are taking X, then successively splitting it into smaller pieces of closed sets. This construction is possible due to the fact that X is separable.

Now we start constructing the continuous map:

For each $a = \langle a_k : k \in \mathbb{N} \rangle \in \mathcal{N}$, and for each $n \in \mathbb{N}$, let $s_n = \langle a_k : k \leq n \rangle$: i.e. the first n digits of a. Then, for each n, choose a point p_n in C_{s_n}. p_n form a Cauchy sequence, and so converge to a point p. $p \in \bigcap \{C_s : s \subset a\}$; Also, it is easily checked by contradiction that p is the only point in the intersection. We let $f(a) = p$.

Now we check that f is continuous. Becausse $\mathcal{N}$ and X are both Polish spaces and metrizable, It suffices to prove sequential continuity. Take $a_n \to a$, where $a_n, a \in \mathcal{N}$. Then we have that for all $m \in \mathbb{N}$, the first m digits of a_n are eventually the same. Thus we see that from the construction of f, $f(a_n) \to f(a)$, thus f is continuous. □

4 Borel Sets

4.1 Definition of Borel Hierarchy

Definition 1. Letbe a Polish space. A set $A \subset X$ is a **Borel set** if it belongs to the smallest $\sigma-$algebra of subsets ofcontaining all closed sets.

Note that if Polish space X is countable, then every $A \in X$ is Borel, because every set is a countable union of point sets. However, if X is uncountable, then this is not the case (for example, [0,1] and the Vitali set, which we will introduce later).

Given $\mathcal{X}$, a set of subsets of S, let us recall the definition of the smallest $\sigma-$algebra containing $\mathcal{X}$, which we will name $\mathcal{S}$. To get all the sets of $\mathcal{S}$, we will have to take intersections and unions of sets in $\mathcal{X}$, then take intersections and unions of all new sets, then repeat this process.

This gives us an idea of how we can go about constructing Borel sets explicitly and inductively: we create a hierarchy of sets, and at each step, take the intersection and unions of all existing sets.

Definition 2. For each $\alpha < \omega_1$, let us define the collection Σ^0_α and Π^0_α of subsets of X:

$\mathbf{\Sigma^0_1}$ = the collection of all open sets [1];

$\mathbf{\Pi^0_1}$ = the collection of all closed sets;

$\mathbf{\Sigma^0_\alpha}$ = the collection of all sets $A = \bigcup_{n=0}^{\infty} A_n$, where each A_n belongs to $\mathbf{\Pi^0_\beta}$ for some $\beta < \alpha$.

$\mathbf{\Pi^0_\alpha}$ = the collection of all sets $A = \bigcap_{n=0}^{\infty} A_n$, where each A_n belongs to $\mathbf{\Sigma^0_\beta}$ for some $\beta < \alpha$.

Remark.

1. This definition follows the ways in which a Borel set can be constructed from simpler sets; we are indexing over all countable ordinals, not just the finite ones, because Borel setes are closed under countable intersection and union.

2. $\mathbf{\Sigma^0_2}$: The countable union of closed sets, are called the F_σ sets.

   $\mathbf{\Pi^0_2}$: The countable intersection of open sets, are called the G_σ sets. [2]

   They are of particular interest when it comes to Baire Category Theory.

3. In our definitions, A_n need not be distinct, and can be $\emptyset$.

4. There is an equivalent definition of $\mathbf{\Pi^0_\alpha}$: $\mathbf{\Pi^0_\alpha}$ = the collection of all complements of sets in $\mathbf{\Sigma^0_\alpha}$.

5. It is clear by induction on α that the elements of each $\mathbf{\Sigma^0_\alpha}$ and $\mathbf{\Pi^0_\alpha}$ are Borel sets.

6. Each $\mathbf{\Sigma^0_\alpha}$ and $\mathbf{\Pi^0_\alpha}$ are closed under finite unions, finite intersections, and inverse image by continuous functions (i.e. if $A \in \mathbf{\Sigma^0_\alpha}$ in Y, then $f^{-1}(A) \in \mathbf{\Sigma^0_\alpha}$ in X whenever $f : X \to Y$ is a continuous function).

Example 2. Here are some Borel sets, and the hierarchy it belongs to.

The Cantor Set: It is a Borel set, a countable intersection of closed sets, thus it is in $\mathbf{\Sigma^0_1}$.

$\mathbb{Q}$: Take any $q \in \mathbb{Q}$, then create an interval around it. Shrink the interval until we only have that point left. Thus $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}} \bigcap_{n=1}^{\infty} [q - 1/n, q + 1/n]$. Thus it is in $\mathbf{\Sigma^0_2}$.

$\mathbb{R} - \mathbb{Q}$ is the complement of $\mathbb{Q}$ so it's in $\mathbf{\Pi^0_2}$.

[1] Note that the notations are in bold. There is another hierarchy, called 'effective Borel hierarchy' or 'lightface Borel hierarchy' which is different. In this essay, we will be talking about the bold faced version only.

4.2 Properties

Once we have a definition, we can ask ourselves whether it is a *good* definition. We would like this hierarchy to be a tool to describe the Borel sets, so does the union of all the hierarchy form a $\sigma-$algebra? And when taking unions, surely we don't need to take unions over both $\mathbf{\Sigma}$ and $\mathbf{\Pi}$ sets. But is the union over one of them the same as the other?

Lemma 3. *for countable ordinals $\alpha < \beta$, we have:*

$$\mathbf{\Sigma^0_\alpha} \subset \mathbf{\Sigma^0_\beta},\ \mathbf{\Sigma^0_\alpha} \subset \mathbf{\Pi^0_\beta},\ \mathbf{\Pi^0_\alpha} \subset \mathbf{\Pi^0_\beta},\ \mathbf{\Pi^0_\alpha} \subset \mathbf{\Sigma^0_\beta}$$

Proof. We only need to prove the first two of these, the rest follows as for collection of sets $\mathcal{A}$ and $\mathcal{B}$, we have $\mathcal{A} \subset \mathcal{B} \Rightarrow \mathcal{A}^c \subset \mathcal{B}^c$ [2].

But note: if α, β are countable ordinals bigger than 1, then the first two inclusions is obvious by definition. The only special case we have to deal with is when $\alpha = 1$.

This case is easily dealt with: Since every open set is the union of countably many closed sets, we have that $\mathbf{\Sigma^0_1} \subset \mathbf{\Sigma^0_2}$. Similarly, every open set is the intersection of countably many open sets, we have that $\mathbf{\Sigma^0_1} \subset \mathbf{\Pi^0_2}$. □

Corollary 1.

$$\bigcup_{\alpha<\omega_1} \mathbf{\Sigma^0_\alpha} = \bigcup_{\alpha<\omega_1} \mathbf{\Pi^0_\alpha}$$

Proof. $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma^0_\alpha} \subset \bigcup_{\alpha<\omega_1} \mathbf{\Pi^0_\alpha}$, and also $\bigcup_{\alpha<\omega_1} \mathbf{\Pi^0_\alpha} \subset \bigcup_{\alpha<\omega_1} \mathbf{\Sigma^0_\alpha}$ □

Remark. This is a lemma that tells us our definition makes sense, when we take unions.

Lemma 4. $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma^0_\alpha}$ *is a $\sigma-$algebra.*

Proof. The axioms are easily checked. Though of course, when checking closure under countable unions, we are taking a countable list of sets A_i from $\mathbf{\Sigma_{\alpha_i}}$, where $\alpha_i < \omega_1$.

When taking the countable union of A_i, we try to check that $\bigcup_{i=0}^{\infty} \mathbf{\Sigma_{\alpha_i}} = \mathbf{\Sigma_{sup\{\alpha_i\}}}$ is still in $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma^0_\alpha}$, which is equivalent to checking if $sup\{\alpha_i\}$ is still countable, which is equivalent to checking if a countable union of countable sets is countable, which is equivalent to Countable Axiom of Choice.

We're fine then, as we are working with ZF and Countable AC. □

Once we have this lemma, we have that every Borel set is in $\mathbf{\Sigma^0_\alpha}$ for some $\alpha < \omega_1$.

Another question about the hierarchy that naturally arises is do we need so many hierarchies? Are there any two hierarchies that are the same? Why does the hierarchy not stop at some countable ordinal, i.e. is it possible to exhaust all the Borel sets with a countable hierarchy?

[2] Note how this is different from the following assertion: when A, B are sets, then $A \subset B \Rightarrow A^c \supset B^c$

We show below that for each $\alpha < \omega_1$, we have: $\mathbf{\Sigma^0_\alpha} \not\subset \mathbf{\Pi^0_\alpha}$, and hence $\mathbf{\Sigma^0_\alpha} \neq \mathbf{\Sigma^0_{\alpha+1}}$ for all $\alpha < \omega_1$, which shows we cannot exhaust all the Borel sets with a countable hierarchy.

First of all, Lemma 2 tells us that there is a continuous mapping from $\mathcal{N}$ onto X. If we can prove $\mathbf{\Sigma^0_\alpha} \not\subset \mathbf{\Pi^0_\alpha}$ for $\mathcal{N}$, then as $\mathbf{\Sigma^0_\alpha}$, $\mathbf{\Pi^0_\alpha}$ are stable under inverse images of continuous functions, we have proven it for X.

We can prove this by exhibiting a set $A \subset \mathcal{N}$ that is in $\mathbf{\Sigma^0_\alpha}$ but not in $\mathbf{\Pi^0_\alpha}$, aka $A \in \mathbf{\Sigma^0_\alpha}$, but $A^c \notin \mathbf{\Sigma^0_\alpha}$.

We start with the following lemma which will give us a property of $\mathbf{\Sigma^0_\alpha}$ to help us construct the set we want.

Lemma 5. *For each $1 \leq \alpha < \omega_1$, there exists a set $U \subset \mathcal{N}^2$ such that U is $\mathbf{\Sigma^0_\alpha}$ (in $\mathcal{N}^2$), and that for every $\mathbf{\Sigma^0_\alpha}$ set $A \in \mathcal{N}$, there exists some $a \in \mathcal{N}$ such that*

$$A = \{x : (x, a) \in U\} \tag{1}$$

We call such a set U a 'universal $\mathbf{\Sigma^0_\alpha}$' set. Further, each $a \in \mathcal{N}$, $\{x : (x, a) \in U\}$ is a $\mathbf{\Sigma^0_\alpha}$ set.

Proof. To construct a universal open set in $\mathcal{N}^2$, a good idea is to start with the basis, build U such that Property 1 holds, and see if it holds for any $\mathbf{\Sigma^0_\alpha}$ set A. Remember the basis $O(s)$ of $\mathcal{N}$, and note that there's a countable number of them. Let us enumerate them as $G_1, ...G_k, ...$ and let $G_0 = \emptyset$.

Now, let

$$(x, y) \in U \iff x \in G_{y(n)} \text{ for some } n.$$

It is easily seen that $U = \bigcup_{n=0}^{\infty} H_n$ where each $H_n = \{(x, y) : x \in G_{y(n)}\}$.

Claim. H_n are open sets.

Proof. Fix $(x, y) \in H_n$, and suppose ϵ, δ are such that $B(x, \epsilon) \subset G_{y_n}$, and $\delta < 1/n$, then $\forall z \in B(y, \delta)$, we have $z(n) = y(n)$, so $B(x, \epsilon) \times B(y, \delta)$ is an open neighbourhood around (x, y). □

Thus U is open, and Property 1 is satisfied for the basic sets G_i.

Now we shall prove this for every open set: suppose G is an open set in $\mathcal{N}$, then we let $a \in \mathcal{N}$ be such that $G = \bigcup_{n=0}^{\infty} G_{a(n)}$; then $G = \{x : (x, a) \in U\}$. Further, we note that given any $a \in \mathcal{N}$, $\{x : (x, a) \in U\} = \bigcup_{n=0}^{\infty} G_{a(n)}$ is an open set.

This prompts us to try induction on α. Our inductive hypothesis is we have a $\mathbf{\Sigma^0_\alpha}$ set U such that:

1. For every $\mathbf{\Sigma^0_\alpha}$ set A, there exists some $a \in \mathcal{N}$ such that $A = \{x : (x, a) \in U\}$;
2. Given $a \in \mathcal{N}$, $\{x : (x, a) \in U\}$ is a $\mathbf{\Sigma^0_\alpha}$ set;

Suppose we have U, a universal $\mathbf{\Sigma^0_\alpha}$ set, and now let us construct a universal $\mathbf{\Sigma^0_{\alpha+1}}$ set V.

First, let us consider a continuous mapping f of $\mathcal{N}$ onto the product space $\mathcal{N}^\omega$. The proof of the existence of such a function can be found in the Appendix.

Once we have the continuous mapping $f : \mathcal{N} \to \mathcal{N}^\omega$, we let

$$(x, y) \in V \iff \text{for some } n, \ (x, y_{(n)}) \notin U.$$

We have $V = \bigcup_{n=0}^\infty H_n$, where each $H_n = \{(x, y) : (x, y_{(n)}) \notin U\}$.

Claim. H_n is a $\mathbf{\Pi^0_\alpha}$ set, thus V is $\mathbf{\Sigma^0_{\alpha+1}}$.

Proof. Let us define a map $g : \mathcal{N}^2 \to \mathcal{N}^2$, given by: $g((x, y)) = (x, y_{(n)})$. H_n^c is the preimage of U under g.

Now, $y \mapsto y_{(n)}$ is the composition of continuous map f and the projection map, therefore it is continuous. So g is continuous, U is $\mathbf{\Sigma^0_\alpha}$, therefore H_n^c is also $\mathbf{\Sigma^0_\alpha}$. The claim follows. □

Now we just have to show V is universal. If A is a $\mathbf{\Sigma^0_{\alpha+1}}$ set in $\mathcal{N}$, then $A = \bigcup_{n=0}^\infty A_n$ where each A_n is $\mathbf{\Pi^0_\alpha}$, and consequently A_n^c is $\mathbf{\Sigma^0_\alpha}$. Thus for each n we can find a_n be such that $\mathcal{N} - A_n = \{x : (x, a_n) \in U\}$.

Let a be such that $a_{(n)} = a_n$ for all n, then $a = \langle a_n \rangle_{n=1}^\infty$ satisfies $A = \{x : (x, a) \in V\}$. Conversely, given $\{x : (x, a) \in V\} = \bigcup_{n=0}^\infty A_n$ where $A_n = \{x : (x, a_{(n)}) \notin U\}$. But A_n is $\mathbf{\Pi^0_\alpha}$, by a similar argument to the previous claim, so A is $\mathbf{\Sigma^0_{\alpha+1}}$.

We have dealt with the successor case, now let us deal with the limit case. Let α be a limit ordinal, and let U_β $(1 \le \beta \le \alpha)$ be universal $\mathbf{\Sigma^0_\alpha}$ sets. Let $1 \le \alpha_0 \le \alpha_1 < \ldots < \alpha_n < \ldots$ be an increasing sequence of ordinals such that $\lim_{n\to\infty} \alpha_n = \alpha$. Let

$$(x, y) \in U \iff \text{for some } n, \ (x, y_{(n)}) \notin U_{\alpha_n}$$

The set U is $\mathbf{\Sigma^0_\alpha}$, again by an argument similar to the previous *Claim*.

If A is a $\mathbf{\Sigma^0_\alpha}$ set in $\mathcal{N}$, then we can write $A = \bigcup_{n=0}^\infty A_n$ where each A_n is $\mathbf{\Pi^0_{\alpha_n}}$ [3]. For each n, let a_n be such that $\mathcal{N} - A_n = \{x : (x, a_n) \in U_{\alpha_n}\}$ [4], and let a be such that $a_{(n)} = a_n$ for all n. Such an a exists, because the map f which we have defined before is onto. Then $A = \{x : (x, a) \in V\}$. Conversely, it is easily checked that $\{x : (x, a) \in V\}$ is $\mathbf{\Sigma_{\alpha+1}}$. □

Remark.

1. This shows that all of the Borel sets can be indexed by $\mathcal{N}$. This also means that the set of Borel sets have cardinality less than or equal to $|\omega^\omega| = 2^{\aleph_0}$.

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[3] This can be done as follows: write $A = \bigcup_{n=0}^\infty B_n$ where $B_n \in \Pi^0_{\beta_n}$. Then, we create A_n by going down the list of B_n and adding empty sets. For example, if α_m $(m > 0)$ is the smallest ordinal such that $B_1 \in \Pi^0_{\alpha_m}$, then let $A_0, \ldots A_{m-1} = \emptyset$, and $A_m = B_1$.

[4] Beware the difference between a and α.

2. In $\mathbb{R}$, we can say more about the cardinality: intervals $[-\infty, r]$ in $\mathbb{R}$ are all Borel, and the set of such intervals cardinality $|\mathbb{R}| = 2^{\aleph_0}$. This means the set of Borel sets have cardinality $2^{\aleph_0}$.

3. This gives a proof for the existence of non-Borel sets in $\mathbb{R}$. The cardinality of $\mathbb{R}$ is $2^{2^{\aleph_0}}$ which is strictly bigger than $2^{\aleph_0}$, thus there exist sets which are non-Borel.

Corollary 2. *For every $\alpha \geq 1$, there is a set $A \subset \mathcal{N}$ that is $\mathbf{\Sigma}^{\mathbf{0}}_{\alpha}$ but not $\mathbf{\Pi}^{\mathbf{0}}_{\alpha}$*

Proof. Let $U \subset \mathcal{N}^2$ be a universal $\mathbf{\Sigma}^{\mathbf{0}}_{\alpha}$ set. Let us consider the set

$$A = \{x : (x, x) \in U\}. \tag{2}$$

A is a $\mathbf{\Sigma}^{\mathbf{0}}_{\alpha}$ set because it is the preimage of U of the continuous map $x \mapsto (x, x)$. If A were also $\mathbf{\Pi}^{\mathbf{0}}_{\alpha}$, then its complement would be $\mathbf{\Sigma}^{\mathbf{0}}_{\alpha}$, and by the definition of a universal set, there is some a such that

$$A = \{x : (x, a) \in U\}^c = \{x : (x, a) \notin U\}. \tag{3}$$

Now, consider a. If $(a, a) \in U$, then $a \in A$ by expression (2), but then $(a, a) \notin U$ by expression (3). Contradiction.

Thus $(a, a) \notin U$. But then by (3) $a \in A$, and by (2) $(a, a) \in U$. Contradiction. □

5 Analytic Set

While Borel sets of reals is closed under Boolean operations, countable unions and preimages of continuous functions, it is NOT closed under continuous images.

In this section, we shall investigate the continuous images of Borel sets.

Definition 3. A subset of a Polish space X is *analytic* if there exists a continuous function $f : \mathcal{N} \to X$ such that $A = f(\mathcal{N})$.

Remark. Because of Lemma 2, we see that $A \subset X$ is analytic if it is the continuous image of a Polish space. It is a generalisation of a the Borel set, and we will prove in the next lemma that they are precisely the continuous images of Borel sets, the objects we want to study.

Definition 4. The projection of a set $S \subset X \times Y$ (into X) is the set $P = \{x \in X : \exists y \ (x, y) \in S\}$.

The following lemma gives equivalent definitions of analytic sets:

Lemma 6. *Let A be a set in a Polish space X. The following are equivalent:*

(i) A is the continuous image of $\mathcal{N}$;

(ii) A is the continuous image of a Borel set B (in some Polish space Y);

(iii) A is the projection of a closed set in $X \times \mathcal{N}$.

Proof. Let us prove two claims first, which will help us with the lemma.

Claim. Every closed set in any Polish space is analytic.

Proof. Note that every closed set of a Polish space is itself a Polish space with respect to the subset topology, and thus a continuous image of $\mathcal{N}$ by Lemma 2. □

Claim. Every Borel set is the projection of a closed set in $X \times \mathcal{N}$.

Proof. To prove claim for every Borel set, it is enough to show that the family P of all subsets of X that are such projections contain all closed sets, all open sets, and is closed under countable unions and intersections.

Because projection map is open, P contains all closed sets. Moreover, every open set is a countable union of closed set, so it suffices to show that P is closed under $\bigcup_{n=0}^{\infty}$ and $\bigcap_{n=0}^{\infty}$. We will prove this using the continuous mapping $a \mapsto \langle a_{(n)} : n \in \mathbb{N} \rangle$ of $\mathcal{N}$ onto $\mathcal{N}^{\omega}$ from Lemma 5.

$\underline{\bigcup_{n=0}^{\infty} A_n}$

For each n, let $F_n \subset X \times \mathcal{N}$ be a closed set such that A_n are the projections of F_n:

$$A_n = \{x : \exists a \ (x, a) \in F_a\}.$$

We want to show that $\bigcup_{n=0}^{\infty} A_n$ is a projection of a closed set in $X \times \mathcal{N}$. To get at such a set, we look at an element $x \in \bigcup_{n=0}^{\infty} A_n$:

$$\begin{aligned} x \in \bigcup_{n=0}^{\infty} A_n &\iff \exists n \ \exists a \ (x, a) \in F_n \\ &\iff \exists a \ \exists b \ (x, a) \in F_{b(0)} \\ &\iff \exists c \ (x, c_{(0)}) \in F_{c_{(1)}(0)} \end{aligned}$$

Recall the use of continuous mapping $f : \mathcal{N} \to \mathcal{N}^{\omega}$ given in Lemma 5 to define $c_{(n)}$. The existence of c comes from the surjectivity of f.

Hence, $\bigcup_{n=0}^{\infty} A_n$ is the projection of the set

$$\{(x, c) : (x, c_{(0)}) \in F_{c_{(1)}(0)}\}$$

This set is closed due to the continuity of f.

$\underline{\bigcap_{n=0}^{\infty} A_n}$

Similar to the previous case, we have:

$$
\begin{aligned}
x \in \bigcap_{n=0}^{\infty} A_n &\iff \forall n\, \exists a\ (x,a) \in F_n \\
&\iff \exists c\, \forall n\ (x, c_{(n)}) \in F_n \\
&\iff \exists c\ (x,c) \in \bigcap_{n=0}^{\infty} \{(x,c) : (x, c_{(n)}) \in F_n\}.
\end{aligned}
$$

And $\bigcap_{n=0}^{\infty} A_n$ is the projection of an intersection of closed sets. □

Once we have these two results the lemma becomes easy, because we have the following corollaries:

$(i) \iff (iii)$ Every closed set in $X \times \mathcal{N}$ is analytic and projection is continuous; Conversely, if $A \subset X$ is an analytic set, $A = f(\mathcal{N})$, then A is the projection of the set $\{(f(x), x) : x \in \mathcal{N}\}$ which is a closed set in $X \times \mathcal{N}$.

$(i) \iff (ii)$ Every Borel set is a projection of a closed set in $X \times \mathcal{N}$. Thus every Borel set is analytic. Conversely, every analytic set is the projection of a closed set in $X \times \mathcal{N}$, which is the continuous image of a Borel set.

□

Now we have an idea of what analytic sets look like, in particular all the Borel sets are analytic.

Later on in this essay, we will prove that the Analytic sets are Lebesgue measurable, and this gives us some examples of sets that are not analytic (e.g. Vitali sets, which we will define later); We will also prove that there exist a set in $\mathcal{N}$ which is analytic, but not Borel.

6 Suslin Operator $\mathcal{A}$

In this section, we use Suslin operator to describe the analytic sets more concretely.

In 1917, Suslin discovered an error in a proof of Lebesgue's article, and it led to a construction of an analytic non-Borel set and the introduction of the operation $\mathcal{A}$.

Before we start, recall that for each $a \in \omega^\omega$, $a|_n$ is the finite sequence $\langle a_k : k < n \rangle$. Recall that for each $s \in Seq$, $O(s)$ is the basic open set $\{a \in \mathcal{N} : a|_n = s\}$ of the Baire space. $O(s)$ is both open and closed. For every set A in a Polish space, $\overline{A}$ denote the closure of A.

Now define Suslin Operator $\mathcal{A}$, which constructs a set from a collection of sets indexed by elements of *Seq*. Let $\{A_s :\ s \in Seq\}$ be such a collection. We define

$$
\mathcal{A}\{A_s :\ s \in Seq\} = \bigcup_{a \in \omega^\omega} \bigcap_{n=0}^{\infty} A_{a|_n}
$$

Remark.

1. If $\{B_s : s \in Seq\}$ is arbitrary, then

$$\bigcup_{a\in\omega^\omega} \bigcap_{n=0}^{\infty} B_{a|_n} = \bigcup_{a\in\omega^\omega} \bigcap_{n=0}^{\infty} (B_{a|_0} \cap B_{a|_1} \cap \ldots \cap B_{a|_n})$$

   Hence $\mathcal{A}\{B_s : s \in Seq\} = \mathcal{A}\{A_s : s \in Seq\}$ where the sets A_s are finite intersections of the sets B_s and satisfy the following condition:

   If $s \subset t$ then $A_s \supset A_t$.

   Thus we shall restrict our use of $\mathcal{A}$ to families that satisfy the above condition.

2. It is easy to see that $\bigcup_{n=0}^{\infty}$, $\bigcap_{n=0}^{\infty}$ are special cases of the Suslin operator.

Lemma 7. *A set A in a Polish space is analytic iff A is the result of operation $\mathcal{A}$ applied to a family of closed sets.*

Proof. First, we show that if $F_s, s \in Seq$, are closed sets in Polish space X, then $A = \mathcal{A}\{F_s : s \in Seq\}$ is analytic. We have

$$\begin{aligned} x \in A &\iff \exists a \in \mathcal{N}\ x \in \bigcap_{n=0}^{\infty} F_{a|_n} \\ &\iff \exists a\ (x, a) \in \bigcap_{n=0}^{\infty} B_n \end{aligned}$$

Where $B_n = \{(x, a) : x \in F_{a|_n}\}$. Now, $B_n = \bigcup_{s\in Seq} F_s \times O(s)$, so each B_n is a Borel set in $X \times \mathcal{N}$ and hence A is analytic.

Conversely, let $A \subset X$ be analytic. There is a continuous function $f : \mathcal{N} \to X$ such that $A = f(\mathcal{N})$. Notice that for every $a \in \mathcal{N}$,

$$\bigcap_{n=0}^{\infty} f(O(a|_n)) = \{f(a)\}.$$

This can be seen easily by contradiction. Now we are almost done, as we can write $A = \bigcup_{a\in\omega^\omega} f(a) = \mathcal{A}\{f(O(s))\}$. The only problem is that we wish that the A is the result of $\mathcal{A}$ applied to a family of closed sets. So let's try to take $\overline{f(O(s))}$.

Claim.

$$\bigcap_{n=0}^{\infty} \overline{f(O(a|_n))} = \{f(a)\}$$

Proof. $f(a) \in \bigcap_{n=0}^{\infty} \overline{f(O(a|_n))}$ is easy to see.

Now, because $\bigcap_{n=0}^{\infty} f(O(a|_n)) = \{f(a)\}$, and the fact that Polish space is homeomorphic to a metric space, we equip it with a metric, and we get $\forall\epsilon > 0$, $\exists n$ such that $f(O(a|_n)) \subset B(f(a), \epsilon)$. Then $\overline{f(O(a|_n))} \subset \overline{B(f(a), \epsilon)}$. Thus $\bigcap_{n=0}^{\infty} \overline{f(O(a|_n))} = \{f(a)\}$. □

Thus:

$$A = f(\mathcal{N}) = \bigcup_{a\in\omega^\omega} \bigcap_{n=0}^{\infty} \overline{f(O(a|_n))},$$

and A is the result of the operation $\mathcal{A}$ applied to the closed sets $\overline{f(O(s))}$[5]. □

Remark. In the process of proving Lemma 7, we have gained an explicit form of writing each analytic set. Suppose $A = f(\mathcal{N})$, the intuition is we can 'fill up' $\mathcal{N}$ with $O(s)$, so $A = \mathcal{A}\{f(O(s))\}$.

7 Hierarchy of Projective Sets

It follows from the preceding section that the collection of all analytic sets in Polish space is closed under countable unions and intersections, continuous images, inverse images, and Suslin operations.

If X is an uncountable Polish space, it is NOT the case that the complement of the Analytic set is analytic. in fact, if the complement is analytic, the analytic set is Borel. In this section we will establish exactly that.

7.1 Definition

To start, let us generalise the Borel Hierarchy by defining the following:

Definition 5.

$\mathbf{\Sigma^1_1}$ = the collection of all analytic sets,

$\mathbf{\Pi^1_1}$ = the complements of analytic sets,

$\mathbf{\Sigma^1_{n+1}}$ = the collection of the projections of all $\mathbf{\Pi^1_n}$ sets in $X \times \mathcal{N}$,

$\mathbf{\Pi^1_n}$ = the complements of the $\mathbf{\Sigma^1_n}$ sets in X,

$\mathbf{\Delta^1_n} = \mathbf{\Sigma^1_n} \cap \mathbf{\Pi^1_n}$.

The set belonging to one of $\mathbf{\Sigma^1_n}$ or $\mathbf{\Pi^0_n}$ are called *projective sets*. It is easily seen that for every n, $\mathbf{\Delta^1_n} \subset \mathbf{\Sigma^1_n} \subset \mathbf{\Delta^1_{n+1}}$ and $\mathbf{\Delta^1_n} \subset \mathbf{\Pi^0_n} \subset \mathbf{\Delta^1_{n+1}}$.

[5] Note that this sets also satisfy the condition that if $s \subset t$, $\overline{f(O(s))} \supset \overline{f(O(t))}$

7.2 Properties

Similar to what we did for the Borel Hierarchy, we shall show that for every n, there is a $\mathbf{\Sigma^1_n}$ set in $\mathcal{N}$ that is not $\mathbf{\Pi^0_n}$, thus the above conclusions are proper inclusions.

The following two results has lots of parallel with Lemma 5 and Corollary 2.

Lemma 8. *For each $n \geq 1$, there exists a universal $\mathbf{\Sigma^1_n}$ set in $\mathcal{N}^2$; i.e. a set $U \subset \mathcal{N}^2$ such that U is $\mathbf{\Sigma^1_n}$ and that for every $\mathbf{\Sigma^1_n}$ set A in $\mathcal{N}$, there exists some $v \in \mathcal{N}$ such that*

$$A = \{x : (x, v) \in U\}.$$

Proof. Let h be a homeomorphism of $\mathcal{N} \times \mathcal{N}$ onto $\mathcal{N}$ [6]. If $n = 1$, let V be a universal $\mathbf{\Sigma^0_1}$ set; if $n > 1$, let V be, by the induction hypothesis, a universal $\mathbf{\Sigma^1_{n-1}}$ set. Let

$$(x, y) \in U \iff (\exists a \in \mathcal{N})\ (h(x, a), y) \notin V. \tag{4}$$

The set $S = \{(x, y, a) : (h(x, a), y) \notin V\}$ is the preimage of V^c under the continuous map $(x, y, a) \mapsto (h(x, a), y)$, so it is closed (if $n = 1$) or $\mathbf{\Pi^1_{n-1}}$ (If $n > 1$). Thus U, being the projection of a closed or $\mathbf{\Pi^1_{n-1}}$ space, is $\mathbf{\Sigma^1_n}$.

Now we check that U is universal $\mathbf{\Sigma^1_n}$.

If $A \subset \mathcal{N}$ is $\mathbf{\Sigma^1_n}$, then by definition there is a closed (or $\mathbf{\Pi^1_{n-1}}$) set B such that

$$x \in A \iff \exists a \in \mathcal{N}\ (x, a) \in B.$$

Since h is a homeomorphism, the set $C = \mathcal{N} - h(B)$ is open (or $\mathbf{\Sigma^1_{n-1}}$) in $\mathcal{N}$. Since V is universal, there exists a v such that $C = \{u : (u, v) \in V\}$. Then by Property 4, we have

$$\begin{aligned} x \in A &\iff (\exists a \in \mathcal{N})(x, a) \in B \iff (\exists a \in \mathcal{N})h(x, a) \notin C \\ &\iff (\exists a \in \mathcal{N})(h(x, a), v) \notin V \iff (x, v) \in U. \end{aligned}$$

□

Hence U is a universal $\mathbf{\Sigma^1_n}$ set.

Corollary 3. *For each $n \geq 1$, there is a set $A \subset \mathcal{N}$ that is $\mathbf{\Sigma^1_n}$ but not $\mathbf{\Pi^0_n}$.*

Proof. Let $U \subset \mathcal{N}^2$ be a universal $\mathbf{\Sigma^1_n}$ set, and let

$$A = \{x : (x, x) \in U\}.$$

A is $\mathbf{\Sigma^1_n}$, because it is the preimage under the continuous map $x \mapsto (x, x)$ of the $\mathbf{\Sigma^1_n}$ set U.

Now, suppose A is $\mathbf{\Pi^0_n}$, then its complement is $\mathbf{\Sigma^0_n}$, and there exists an a such that

$$A = \{x :\ (x, a) \notin U\}.$$

which leads to a contradiction, in the same way as in Corollary 2. □

[6] For explicit construction of the homeomorphism, see Appendix.

Corollary 4. *There exists a set in $\mathcal{N}$ which is analytic, but not Borel.*

Proof. Let $U \subset \mathcal{N}^2$ be a universal analytic set (i.e. a universal Σ_1^1 set), and let

$$A = \{x \in \mathcal{N} : \ (x, x) \in U\}$$

A is analytic, by the same argument as before. If A is Borel, then so is A^c, thus A^c is analytic, and there exists $a \in \mathcal{N}$ such that

$$A = \{x : \ (x, a) \notin U\}$$

We reach a contradiction, in the same way as in Corollary 2. □

7.3 $\mathbf{\Sigma_1^1}$−Separation

The collection of all $\mathbf{\Delta_1^1}$ sets in a Polish space is a σ−algebra, and contains all Borel sets. It turns out, $\mathbf{\Delta_1^1}$ is exactly the collection of all Borel sets.

Theorem 1 (Suslin)**.** *Every analytic set whose complement is also analytic is a Borel set. Thus $\mathbf{\Delta_1^1}$ is the collection of all Borel sets.*

In order to prove this theorem, we will prove the $\mathbf{\Sigma_1^1}$−Separation theorem, from which the theorem follows easily.

Letbe a Polish space, and let A and B be two disjoint analytic sets in X. We say that A and B are separated by a Borel set if there exists a Borel set D such that $A \subset D$ and $B \subset X - D$.

Lemma 9. *Any two disjoint analytic sets are separated by a Borel set.*

Remark. This lemma is often called *the $\mathbf{\Sigma_1^1}$−Separation Principle*. It implies Suslin's theorem since if A is an analytic set such that $B = X - A$ is also analytic, A and B are also separated by a Borel set D and we clearly have $D = A$.

Proof. First we make the following observation: If $A = \bigcup_{n=0}^{\infty} A_n$ and $B = \bigcup_{n=0}^{\infty}$ are such that for all n, m, A_n and B_m are separated, then A and B are separated. This is proved as follows: For each n and each m, let $D_{n,m}$ be a Borel set such that $A_n \subset D_{n,m} \subset X - B_m$. Then A and B are separated by the Borel set $D = \bigcup_{n=0}^{\infty} \bigcap_{m=0}^{\infty} D_{n,m}$.

Let A, B be two disjoint analytic sets in X. Let f, g be continuous functions such that $A = f(\mathcal{N})$ and $B = g(\mathcal{N})$. For each $s \in Seq$, let $A_s = f(O(s))$ and $B_s = g(O(s))$; the sets A_s and B_s are all analytic sets. For each s, we have $A_s = \bigcup_{n=0}^{\infty} A_{s\frown n}$ and $B_s = \bigcup_{m=0}^{\infty} B_{s\frown m}$. If $a \in \omega^\omega$, then

$$\{f(a)\} = \bigcap_{n=0}^{\infty} f(O(a|_n)) = \bigcap_{n=0}^{\infty} A_{a|_n},$$

and similarly for the sets B_s.

Let $a, b \in \omega^\omega$ be arbitrary. Since $f(\mathcal{N})$ and $g(\mathcal{N})$ are disjoint, we have $f(a) \neq g(b)$. Let G_a and G_b be two disjoint open neighbourhoods of $f(a)$ and

$g(b)$, respectively (which is possible to find, as Polish spaces are Hausdorff). By the continuity of f and g there exists some n such that $A_{a|_n} \subset G_n$ and $B_{b|n} \subset G_b$. It follows that fixing a, b, there exists n such that the sets $A_{a|_n}$ and $B_{b|n}$ are separated by a Borel set.

We shall now show, by contradiction, that the sets A and B are separated by a Borel set. If A and B are not separated, then because $A = \bigcup_{n=0}^{\infty} A_{\langle n \rangle}$ and $B = \bigcup_{m=0}^{\infty} B_{\langle n \rangle}$, there exists n_0 and m_0 such that $A_{\langle n_0 \rangle}$ and $B_{\langle m_0 \rangle}$ are not separated.

The similarly there exists n_1 and m_1 such that the sets $A_{\langle n_0, n_1 \rangle}$ and $B_{\langle m_0, m_1 \rangle}$ are not separated, and so on. In other words, there exists $a = \langle n_0, n_1, ... \rangle$ and $b = \langle m_0, m_1, ... \rangle$ such that for every k, $A_{\langle n_0, ... n_k \rangle}$ and $B_{\langle m_0, ... m_k \rangle}$ are not separated. This is a contradiction, since in the preceding paragraph we proved exactly the opposite: There is a k such that $A_{a|k}$ and $B_{b|k}$ are separated. $\square$

8 Properties

We will next look into additional properties of analytic sets including measurability and the Baire Property. In this section, we will go over these concepts.

8.1 Lebesgue Measure

The Lebesgue measure is a way of measuring the size of a set, the intuitive way of thinking is that Lebesgue measure of a cube is just its volume. Let us use $v(I)$ to define the volume of I. Of course, when we start to generalise v, we would ran into some problems, for example, how would we define the volume of a Cantor set?

The standard way of defining Lebesgue measure is to define first the *outer measure* $\mu^*(X)$ of a set $X \subset \mathbb{R}^n$ as the infimum of all possible sums $\Sigma\{v(I_k) : k \in \mathbb{N}\}$ where $\{I_k : k \in \mathbb{N}\}$ is a collection of $n-$dimensional intervals such that $X \subset \bigcup_{k=0}^{\infty} I_k$. We have $\mu^* \geq 0$ and X is null if $\mu^*(X) = 0$.

A set $A \subset \mathbb{R}^n$ is Lebesgue measurable if for each $X \subset \mathbb{R}^n$,

$$\mu^*(X) = \mu^*(X \cap A) + \mu^*(X - A).$$

For a measurable set A, we write $\mu(A)$ instead of $\mu^*(A)$, and call $\mu(A)$ the Lebesgue measure of A.

Lebesgue measure have the following properties, whose proofs we will omit:

(i) Every cuboid is Lebesgue measurable, and its measure is its volume.

(ii) The Lebesgue measure sets form a $\sigma-$algebra, hence every Borel set is measurable.

(iii) μ is $\sigma-$additive: If $\{A_n\}$ are countable pairwise disjoint and measurable sets, then

$$\mu(\bigcup_{n=0}^{\infty} A_n) = \Sigma_{n=0}^{\infty} \mu(A_n).$$

(iv) μ is $\sigma-$finite: If A is measurable, then there exist measurable sets A_n, $n < \omega$ such that $A = \bigcup_{n=0}^{\infty} A_n$ and $\mu(A_n) < \infty$ for each n.

Now let us see the following lemma that will be very important later in the proof of Theorem 3:

Lemma 10. *For any set $X \subset \mathbb{R}^n$ there exists a measurable set $A \supset X$ with the property that whenever $Z \subset A - X$ is measurable, then Z is null.*

Proof. Note that we are not assuming that X is measurable, so let us look at its outer measure $\mu^*(X)$, defined by $\mu^*(X) = inf\{\mu(A)$: A is measurable and $A \supset X\}$.

If $\mu^*(X) < \infty$, then there is a measurable $A \supset X$ such that $\mu(A) = \mu^*(X)$; this A meets the requirement.

If $\mu^*(X) = \infty$, there exists pairwise disjoint X_n such that $X = \bigcup_{n=0}^{\infty} X_n$ and that for each n, $\mu^*(X_n) < \infty$. This can be done for example simply by dividing $\mathbb{R}^n$ into countable number of cubes, number them, and let X_n be the intersection of X and the nth cube.

Now we have reduced this to the previous case. Let $A_n \supset X_n$, $n < \omega$, be measurable sets such that $\mu(A_n) = \mu^*(X_n)$ and let $A = \bigcup_{n=0}^{\infty} A_n$. □

8.1.1 Examples

Not every set is measurable. A typical counter-example is the Vitali set. It is not measurable implies it is not Borel either.

Definition 6 (Vitali Set)**.** Take $[0, 1]$, and define an equivalence relationship by the following: for $x, y \in [0, 1]$, $x \sim y$ iff $(x - y) \in \mathbb{Q}$.

For each equivalence class, take a class representative, and let S be the set of such representatives.

Claim. The Vitali set is not measurable.

Proof. We will show that this set is not measurable.

First of all, note that for any $u \in [0, 1], \exists! v \in S$ such that $v \sim u$, aka $(v - u) \in \mathbb{Q}$. Furthermore, for any $v \in S$, $q \in \mathbb{Q}$, $u + q \notin S$. Thus we see that for each $q \in \mathbb{Q}$, set $S + q$ is disjoint to S, also $[0, 1] = \bigcup_{q \in \mathbb{Q}}(S + q)$.

Now, suppose that S is measurable, with measure m, then this says $[0, 1] = \Sigma_{q \in \mathbb{Q}} |S|$, which is either 0 or ∞. Contradiction. □

8.2 The Property of Baire

Let us consider a Polish space X. Let us call a set $A \subset X$ *nowhere dense* if the complement of A contains a dense open set. This means for any open set G, $A \cap G$ is not dense in G. Not explicitly, there is $H \subset G$ such that $A \cap H = \emptyset$.

A set $A \subset X$ is *meagre* (or of *first category*) if A is a union of countable many nowhere dense sets. A nonmeagre set is called a set of *second category.*

Lemma 11. *The subset of a meager set is meager.*

Proof. Suppose A is meager, and $B \subset A$. Then $A = \bigcup_{n=0}^{\infty} U_i$, where U_i are nowhere dense. Then $B = \bigcup_{n=0}^{\infty} U_i \cap B$, and $U_i \cap B$ is also nowhere dense, thus B is meager. $\square$

A fundamental result in analysis is the Baire Category Theorem, which we will state here and not prove.

Theorem 2 (Baire Category Theorem)**.** *In Polish space, every nonempty open set is of nonmeagre.*

Definition 7 (Baire Set)**.** Given a Polish space X, a set $A \subset X$ is a Baire Set if there exists an open set G such that $G \triangle A$ is meager.

Remark. The notion of a Baire topological space is completely different from that of Baire set.

Lemma 12. *The sets having the Baire property form a $\sigma-$algebra, hence every Borel set has the Baire property.*

Proof. All the open sets are Baire, in particular X is Baire.

It is also easy to see that the union of countably many sets with Baire property has the Baire property.

Now let's check complements. Note that if G is open, then $\overline{G} - G$ is nowhere dense. Hence if $A \triangle G$ is meager then $(X - A) \triangle (X - \overline{G}) = A \triangle \overline{G}$ is meager, and $X - \overline{G}$ is open, so it follows that the complement of a set with the Baire property also has the Baire property.

Thus the Baire sets are a $\sigma-$algebra, and contains all the open sets, as a result it contains all the Borel sets. $\square$

Lemma 13. *For any set S is a Polish space X, there exists a set $A \supset S$ that has the Baire property, and such that whenever $Z \subset A - S$ has the Baire property, then Z is meagre.*

Remark. Compare this lemma to Lemma 10.

Proof. Let us consider a fixed countable topology basis O for X. Let $S \subset X$. Let

$$D(S) = \{x \in X : \text{ for every } U \in \mathcal{O} \text{ such that } x \in U,\ U \cap S \text{ is not meager } \}.$$

Note that the complement of $D(S)$ is the union of $U \in \mathcal{O}$ such that $U \cap S$ is meager. Thus $D(S)^c$ is open. So $D(S)$ is closed.

the union of open sets and hence open; thus $D(S)$ is closed.

The set $S - D(S)$ is the union of all $S \cap U$ where $U \in O$ and $S \cap U$ is meagre; since $\mathcal{O}$ is countable, $X - D(S)$ is a countable union of meager sets, so $S - D(S)$ is meager. Let

$$A = S \cup D(S).$$

Since $A = (S - D(S)) \cup D(S)$ is the union of a meager and a closed set, A has the Baire property. This is because the Baire sets form a $\sigma-$algebra which includes Borel sets and the meager sets, so A, the union of a Borel and meager set, is a Baire set.

Let $Z \subset A - S$ have the Baire property: we shall show that Z is meager.

Because Z is Baire, we see that there is an open set G such that $G \triangle Z$ is meager. The subset of a meager set is meager, and as a result $G - Z$ is meager. Because $\mathcal{O}$ is the basis of the Polish space X, G is the union of open sets in this basis, and we can find $U \in \mathcal{O}$ such that $U - Z$ is meager.

Suppose, for contradiction, that Z is non-meager. Then $U \cap Z \neq \emptyset$. If not then $U \triangle Z = U \cup Z$ is non-meager.

Because $Z \subset A - S$, we have that $U \cap S \subset U - Z$, so $U \cap S$ is meager.

Also, $Z \subset D(S)$, so there exist $x \in U$ such that $x \in D(S)$, this $U \cap S$ is non-meager, a contradiction.

Otherwise there is $U \in O$ such that $U - Z$ is meagre; hence $U \cap S$ is meager.

Since $U \cap Z \neq \emptyset$ and $Z \subset D(S)$, there is $x \in U \cap Z$ such that $x \in D(S)$, and hence $U \cap S$ is not meager by the definition of $D(S)$, a contradiction. □

8.2.1 Examples

Not every set is a Baire set, one example is the familiar Vitali set.

Claim. The Vitali set does not have the Baire property.

Proof. First of all, note that 'meagre' and 'Baire' property are invariant under translation.

Let S be the Vitali set. If S has the Baire property, then there is an open set G such that $G \triangle S$ is meager. We can find an interval $(a, b) \subset G$, and because the subset of a meager set is still meager, and $(a, b) - S \subset G \triangle S$, we have that $(a, b) - S$ is meager.

Now, we know that for all $q \in \mathbb{Q}$, $S + q \cap S = \emptyset$. $(a, b) \cap (S + q) \subset (a - b) - S$, thus $(a, b) \cap (S + q)$ is meager for all rational $q \neq 0$. Thus $S \cap (a - q, b - q)$ is meager for all rational $q \neq 0$.

Because $S = \bigcup_{q \in \mathbb{Q}} S \cap (a - q, b - q)$, we get S is also meager. However, $[0, 1] = \bigcup_{q \in \mathbb{Q}} S + q$, thus $[0, 1]$ is meager. This is not true, contradiction. □

8.3 A note on 'Smallness'

We have now seen several adjectives that means 'small', like 'null' and 'meagre'. But these adjectives describes sizes in complely different ways. For example, the real line can be decomposed into a null set and a meager set.

Lemma 14. *$\mathbb{R}$ can be decomposed into a null set and a meager set.*

Proof. Let $\mathbb{Q}$ be a enumeration of the rationals. For each $n \geq 1$ and $k \geq 1$, let $I_{n,k}$ be the open interval with center q_n and length $1/(k \cdot 2^n)$. Let $D_k = \bigcup_{n=1}^{\infty} I_{n,k}$ and $A = \bigcap_{k=1}^{\infty} D_k$.

Each D_k is open because it is the union of open sets, it is dense as it contains $\mathbb{Q}$, and $\mu(D_k) \leq 1/k$. Hence A is null, and $\mathbb{R} - A$ is meager. □

Another interesting example is the Cantor set $\mathcal{C}$. $|\mathcal{C}|$ is the continuum, i.e. the same cardinality as the reals. However, the Lebesgue measure of the Cantor set is 0, and the set is meager.

9 Analytic Sets: Measure and Category

Theorem 3.

(i) Every analytic set of reals is Lebesgue measurable.

(ii) Every analytic set has the Baire property.

Proof. (i) Let A be an analytic set of reals (or a subset of $\mathbb{R}^n$). Let $f : \mathcal{N} \to \mathbb{R}$ be a continuous function such that $A = f(\mathcal{N})$. For each $s \in Seq$, let $A_s = f(O(s))$. We have by lemma 7:

$$A = \mathcal{A}\{A_s : s \in Seq\} = \mathcal{A}\{\overline{A_s} : s \in Seq\}, \tag{5}$$

And for every $s \in Seq$,

$$A_s = \bigcup_{n=0}^{\infty} A_{s\frown n}.$$

By lemma 10, there exist for each $s \in Seq$ a measurable set $B_s \supset A_s$ such that every measurable $Z \subset B_s - A_s$ is null. Since $\overline{A_s}$ is measurable, we may actually find B_s such that $A_s \subset B_s \subset \overline{A_s}$.

Let $B = B_\emptyset$. Since B is measurable, to show that A is measurable, it suffices to show that $B - A$ is a null set.

Notice that because $A_s \subset B_s \subset \overline{A_s}$, and because Property 5 holds, we have that

$$A = \mathcal{A}\{B_s : s \in Seq\}.$$

Thus

$$B - A = B - \bigcup_{a\in\omega^\omega} \bigcap_{n=0}^{\infty} B_{a|_n}.$$

We claim that

$$B - \bigcup_{a\in\omega^\omega} \bigcap_{n=0}^{\infty} B_{a|_n} \subset \bigcup_{s\in Seq} (B_s - \bigcup B_{s\frown k}). \tag{6}$$

To prove Property 6, we assume that $x \in B$ is such that x is not a member of the right hand side. Then for every s, if $x \in B_s$, then $x \in B_{s\frown k}$ for some k. Hence there is k_0 such that $x \in B_{(k_0)}$, a k_1 such that $x \in B_{\langle k_0,k_1\rangle}$, etc. Let $a = \langle k_0, k_1, ...\rangle$; we have $x \in \bigcap_{n=0}^{\infty} B_{a|_n}$ and hence x is not a member of the left hand side [7].

[7] We have seen this trick of consecutively selecting elements to build an index a before, in the proof of Lemma 9.

Thus we have

$$B - A \subset \bigcup_{a \in Seq} (B_s - \bigcup_{n=0}^{\infty} B_{s \frown k}).$$

Since *Seq* is a countable set, it suffices to show that each $B_s - \bigcup_{k=0}^{\infty} B_{s \frown k}$ is null. Let $s \in Seq$, and let $Z = B_s - \bigcup_{k=0}^{\infty} B_{s \frown k}$. We have:

$$Z = B_s - \bigcup_{k=0}^{\infty} B_{s \frown k} \subset B_s - \bigcup_{k=0}^{\infty} A_{s \frown k} = B_s - A_s.$$

Now, because $Z \subset B_s - A_s$ and because Z is measurable, Z must be null.

(ii) The proof for the second part is almost identical to the proof of (i), except we are using Lemma 13 instead of Lemma 10.

Let A be an analytic set of a Polish space X, and let $f : \mathcal{N} \to X$ be the continuous function such that $f(\mathcal{N}) = A$. For each $s \in Seq$, let $A_s = f(O(s))$. We have by lemma 7:

$$A = \mathcal{A}\{A_s : s \in Seq\} = \mathcal{A}\{\overline{A_s} : s \in Seq\},$$

and for every $s \in Seq$, $A_s = \bigcup_{n=0}^{\infty} A_{s \frown n}$.

By Lemma 13, there exists for each $s \in Seq$ a Baire set $B_s \supset A_s$ such that every Baire set $Z \subset B_s - A_s$ is meager. Since $\overline{A_s}$ is a closed set, thus Baire, we can actually find B_s such that $A_s \subset B_s \subset \overline{A_s}$.

Let $B = B_\emptyset$. Since B is Baire, to show that A is Baire, it suffices to show that $B - A$ is meager.

The rest of the proof proceed in exactly the same way as (i), noting that for each s, $B_s - \bigcup_{k=0}^{\infty} B_{s \frown k}$ is meager. □

10 What Next

The main results of descriptive set theory on Lebesgue measure can be proved in a more general context, namely for reasonable $\sigma-$additive on Polish spaces. An example of such a measure is a product measure in the Cantor space $\{0, 1\}^\omega$.

Another property of the analytic sets that we haven't mentioned is connected to the idea of perfect subsets.

A nonempty closed set is *perfect* if it has no isolated points. It is a very interesting set with lots of properties (see Cantor-Bendixson Theorem) [3], and we could also prove that every uncountable analytic set contains a perfect subset.

References

[1] Thomas Jech, *Set Theory*, Chapter 11

[2] https://en.wikipedia.org/wiki/Borel_hierarchy

[3] https://en.wikipedia.org/wiki/Perfect_set

Appendix

Theorem 4. *There exists a continuous function $f : \mathcal{N} \to \mathcal{N}^\omega$.*

Proof. We want a map $f : \mathcal{N} \mapsto \mathcal{N}^\omega$. Before we start, let's examine $\mathcal{N}^\omega$.

$\mathcal{N}^\omega$ has the product topology. Now, because $\mathcal{N}$ is metrixable, $\mathcal{N}^\omega$ is also mtrizable.

To see this, equipt $\mathcal{N}$ with the metrix d_1, and when $x = \langle x_i \rangle$, $y = \langle y_i \rangle$, we define

$$d_2(x, y) = \Sigma_{i \in \mathbb{N}} 2^{-i} \frac{d_1(x_i, y_i)}{1 + d(x_i, y_1)}$$

This is a metrix on $\mathcal{N}^\omega$ and induced the same topology. Thus to prove $f : \mathcal{N} \to \mathcal{N}^\omega$ is continuous, we just have to prove that it is sequentially continuous with respect to the metric.

Let us present f first, then we prove it's continuous.

Given $a \in \mathcal{N}$, we wish to define its image under the map f. The image of a needs to be a member of $\mathcal{N}^\omega$. Let us define $a_{(n)}$, the nth coordinate of *the image of* a. $a_{(n)} \in \mathcal{N}$, so we need to specify $a_{(n)}(k)$.

Thus to specify $a_{(n)}$ we require a mapping from $\mathcal{N} \times \mathcal{N}$ to $\mathcal{N}$. For this, there is a canonical one to one pairing Γ, as $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. Let $a_{(n)}(k) = a(\Gamma(n, k))$.

So $a_{(n)}$ is a reordering of the coordinates of a using Γ.

Now take $a_n \xrightarrow{d_1} a$, where $a_n, a \in \mathcal{N}$. We will show that $f(a_n) \xrightarrow{d_2} f(a)$. To do that, it is enough to show that the map $a \mapsto a_{(n)}$ is continuous.

Remember the way we defined $a_{(n)} : a_{(n)}(k) = a(\Gamma(n, k))$, which gives a permutation of the coordinates of a. If $a_n \to a$, then a_n 'will start to look very similar to a'. For every $N \in \mathbb{N}$, there exists m such that for all $k > m$, the first N coordinates of $a_{(k)}$ are the same. Thus we can see that $a \mapsto a_{(n)}$ is a continuous map. As a result, f is continuous.

□

Corollary 5. *There exist a homeomorphism from $\mathcal{N}$ to $\mathcal{N}^2$.*

Proof. Very similar to the previous theorem, let us define $f : \mathcal{N} \mapsto \mathcal{N}^2$, where the first coordinate of $f(a)$ is $a_{(1)}$, second coordinate of $f(a)$ is $a_{(2)}$. Then, take Γ, the bijection between $\mathbb{N}^2$ and $\mathbb{N}$, and let $a_{(n)}(k) = a(\Gamma(n, k))$. We have shown that this map is continuous. Because Γ is a bijection, one can also easily see that f is a bijection.

Now we just have to show that f is open. For that, consider the inverse map, $(a_1, a_2) \mapsto a$. Each element of a is again the reshuffling of the elements in a_1 and a_2. Thus by same argument as the previous theorem, we see that it is continuous. Thus f is a homeomorphism. □