The Constructible Universe

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Introduction

When we study Zermelo–Fraenkel set theory there are several questions that naturally arise: we would like to know whether important hypothesis such as the Axiom of Choice (AC) and the General Continuum Hypothesis (GCH) are consistent with the ZF axioms.

Gödel proved the consistency of GCH and AC using the constructible universe L. L has very special properties, for example: L is a model of the ZF axioms; in fact, it is the smallest inner model of ZF, and if we regard L as the whole universe and took its version of L inside of it, we get L again.

Further, inside L, both AC and GCH are true. This is especially important, as this means AC and GCH are both consistent with ZF.

Sources

Most of the material here follow closely to Chapter 12 and 13 in Jech's book *Set Theory*. Professor Löwe kindly gave a different way to prove GCH and AC holds, which is much shorter and very elegant. If there are any mistakes in the proofs it is entirely my fault.

In this essay we are assuming knowledge of basic set theory, and we will be working with ZF axioms throughout.

Outline

In this essay, we will start by establishing some preliminaries and proving some very important background results and introduce constructible sets.

Next, we will check that L is a model of the ZF axioms. Then we will use Gödel Normal Forms to show the axiom of constructibility and that L is the smallest inner model. Finally, we will show that GCH theorems holds in L, and as a result AC also holds.

1 Preliminaries

1.1 Models of Set Theory and Relativization

Before we start laying down the definitions of constructible sets, let us first make clear the concepts of relativization.

The language of set theory consists of one symbol, $\in$. Let M be a class, and let $\phi(x_1, ...x_n)$ be a formula in the language of set theory. Then the *relativization*, of ϕ in M, written as $\phi^M(x_1, ...x_n)$, is defined below [1]:

$$\begin{gathered}(x \in y)^M \Leftrightarrow x \in y,\\ (x = y)^M \Leftrightarrow x = y,\\ (\neg\phi)^M \Leftrightarrow \neg\phi^M,\\ (\phi \ \wedge \ \psi)^M \Leftrightarrow \phi^M \wedge \psi^M,\\ (\exists x \ \phi)^M \Leftrightarrow (\exists x \in M)\phi^M\end{gathered}$$

We can also extend relativization to concepts other than formulae, namely classes, operations and constants:

If C is a class $\{x : \phi(x)\}$ then C^M is the class $\{x \in M : \phi^M(x)\}$.

If c is a constant symbol, and $c \in M$, then c^M is the corresponding constant in M. So $\emptyset^M = \emptyset$ if $\emptyset \in M$, $\omega^M = \omega$ if $\omega \in M$ etc.

1.2 Subformulae

Here we define the notion of subformulae, which will be useful later on. Given a formula ϕ, its subformulae are its building blocks. Its proper subformulae are the subformulae apart from the ϕ itself.

Definition 1.

1. *Atomic Formulae don't have proper subformulae.*

2. *If $\phi = \neg\psi$, $(\forall x)\psi$ or $\exists\psi$, the proper subformulae of ϕ are ψ together with all proper subformulae of ψ.*

3. *If ϕ is $\psi \Rightarrow \chi$, the proper subformulae of ϕ are ψ, χ together with all proper subformulae of ψ and χ.*

1.3 Reflection Principle

Theorem 1 (Reflection Principle)**.** *Let ϕ be a formula. For each M_0, there exists a set $M \supset M_0$ such that*

$$\phi^M(x_1, ...x_n) \iff \phi(x_1, ...x_n)$$

for all $x_1, ...x_n \in M$. We say that M reflects ϕ.

[1] Here, we are assuming that ϕ is a formula that lives in V. The actual concept of relativization is much more general. First of all we assume M is a set with binary relation E, we can get a relativization $\phi^{M,E}$.

Proof. For atomic formulae $x = y$ and $x \in y$ for example, the theorem is obviously true: we can just take M_0. This suggests we could try starting from atomic formulae, and build up to ϕ.Indeed, suppose ϕ and ψ are such that the theorem holds, then the theorem also holds for $\phi \wedge \psi, \neg\phi, \phi \vee \psi, \phi \Leftrightarrow \psi$ and $\phi \Rightarrow \psi$ too. However, it is not so obvious that theorem holds for $(\exists x)\phi$. To prove that it holds for this formula, we need the next lemma:

Lemma 1. *Let $\phi(u_1, ...u_n, x)$ be a formula. For each set M_0, there exists a set $M \supset M_0$ such that*

if $(\exists x)\ \phi(u_1, ...u_n, x)$, then $(\exists x \in M)\ \phi(u_1, ...u_n, x)$ for every $u_1, ...u_n \in M$.

Proof. Let us fix $u_1, ...u_n$ first, and let $C = \{x : \phi(u_1, ...u_n, x)\}$.

Define $\hat{C} = \{x \in C : (\forall z \in C) : \mathrm{rank}(x) < \mathrm{rank}(z)\}$, i.e. 'the smallest elements in C'. This picks out a sample of x that works for this choice of u_i.

Next, for every choice of $u_1, ..., u_n$, let $H(u_1, ...u_n) = \hat{C}$.

Thus $H(u_1, ..u_n)$ is the set with the property

$$\exists x\phi(u_1, ...u_n, x) \Rightarrow (\exists x \in H(u_1, ...u_n))\phi(u_1, ...u_n, x)$$

Now we construct the set M by induction. Step by step, we throw in the set $H(u_1, ...u_m)$ where u_i range over the set we already have.

Explicitly: let $M = \bigcup_{i=0}^{\infty} M_i$, where for all $i \in \omega$,

$$M_{i+1} = M_i \cup \bigcup \{H(u_1, ...u_n) : u_1, ..., u_n \in M_i\}$$

Now, if $u_1, ..., u_n \in M$, then there exists $i \in \omega$ such that $u_1, ..., u_n \in M_i$; and if $\phi(u_1, ...u_n, x)$ holds for some x, then it holds for some $x \in M_{i+1}$ by the construction of M. Thus M is the set we want. $\square$

Once we have this lemma, we can prove this theorem. Let $\phi(x_1, ...x_n)$ be a formula. Let us try to simplify ϕ first.

We may assume that the universal quantifier does not occur in ϕ, because we may replace $\forall x$ with $\neg\exists x\neg$.

Next, let $\phi_1, ...\phi_k$ be all the subformluae of the formula ϕ.

These subformulae are simpler than ϕ in complexity, so we can assume by induction that these subformulae satisfy the theorem. Next we construct ϕ, and the only problematic case would be showing that the theorem holds for $(\exists x)\phi$.

By Lemma 1, there exist $M \supset M_0$ such that for $j = 1, 2, ...k$ and $u... \in M$,

$$(\exists x)\phi_j(u, ...x) \Rightarrow (\exists x \in M)\phi_j(u...x) \tag{1}$$

Now we can easily show that M reflects $(\exists x)\phi_j$: Take $u_1...u_m \in M$, then

$$\begin{aligned} M \models \exists x\ \phi_j(u_1, ...u_m, x) &\Leftrightarrow (\exists x \in M)\ \phi_j^M(u_1, ...u_m, x) \\ &\Leftrightarrow (\exists x \in M)\phi_j(u_1, ...u_m, x) \\ &\Leftrightarrow \exists x\ \phi_j(u_1, ...u_m, x) \end{aligned}$$

We can take away the realization there, because we have restricted u_i to M; the last equivalence is because of (1).

To finish up: M reflects every atomic formulae, and if it reflects ϕ and ψ, then it reflects $\neg\phi, (\exists x)\phi, \phi \wedge \psi, \phi \vee \psi, \phi \Leftrightarrow \psi$ and $\phi \Rightarrow \psi$. So we can build ϕ using these simpler components, and M reflects it. □

Remark.

1. Lemma 1 is very important on its own, it gives a 'representation' of the set of x that makes ϕ true.
2. The intuition behind the proof is to take 'samples': take one element from C, throw it in M, and do this for every other choice of $u_1, ...u_n$. Be careful that after each step we have added more elements.
3. The construction of $\hat{C}$ is known as the 'Scott's Trick', it is used to avoid the problem of Choice.

Corollary 1. *(Variation on Reflection Principle) Let ϕ be a formula. There are arbitrarily large limit ordinals α such that $\phi^L(x_1, ...x_n) \Leftrightarrow \phi^{L_\alpha}(x_1, ...x_n)$ for all $x_1, ...x_n \in L_\alpha$.*

Proof. This is very similar to the proof of the Reflection Principle, we replace M with L_α and perform some modifications. See Appendix, Theorem 8. □

1.4 Collection Principle

Theorem 2 (Collection Principle)**.**

$$(\forall x)(\exists y)(\forall u \in x)[(\exists z)\phi(u, z, p...) \Rightarrow (\exists z \in y)\phi(u, z, p...)] \tag{2}$$

Remark. This theorem is an easy corollary of Lemma 1. It says that given a collection of classes C_u, $u \in x$, then there is a set y such that for every $u \in x$, if $C_u \neq \emptyset$, then $C_u \cap y \neq \emptyset$. I.e. y 'samples' C_u.

Proof. In Lemma 1, take $M_0 = x$. Then we get $y \supset x$ such that

$$(\forall x)(\exists y)(\forall u \in y)[(\exists z)\phi(u, z, p...) \Rightarrow (\exists z \in y)\phi(u, z, p...)] \tag{3}$$

Note that $M \supset x$, which is far bigger than necessary. But that's not a problem, we can just restrict it to x to get the result we want. □

1.5 Δ_0 formulae

If we are trying to find a model for ZF, a good place to start would be the transitive classes. For example something like $V_{\omega+\omega}$: this structure satisfies every ZF axiom, except for the axiom of replacement.

So let's focus our attention to a transitive classes M, and ask ourselves: what sort of formulae would hold in M? For example, the statement 'x is an empty set': if x is empty then x is also empty in M.

After trying to go from V to M, we can also ask ourselves, what can M tell us about V? Is there some formula that, if true in M, is true in V?

This leads us to the following family of formulae, Δ_0, which have the property that if ϕ is Δ_0, then ϕ is true in V iff ϕ is true in M.

Definition 2. *A formula of set theory is a Δ_0 formula if:*

(i) It has no quantifiers.

(ii) It is $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$, $\phi \Rightarrow \psi$ or $\phi \Leftrightarrow \psi$ where ϕ, ψ are Δ_0 formulae.

(iii) It is $(\exists x \in y)\psi$, $(\forall x \in y)\psi$ where $x \in M$, and ϕ, ψ are Δ_0 formulae.

Let us see a few examples of Δ_0 formulae: These will give us good intuition for what things are Δ_0, and also all of them are useful later. Most of these proof are simple reformulations.

Lemma 2. *The following expressions can be written as Δ_0 formulae, and thus absolute for all transitive models:*

(i) $x = \{a, b\}, x = (a, b), x = \emptyset,\ x \subset y$

(ii) x *is transitive,* x *is an ordinal,* $x = \omega$

(iii) $Z = X \times Y, Z = X - Y, Z = X \cap Y, Z = \bigcup X$

(iv) $Z \in \operatorname{dom}(X),\ Z = \operatorname{dom}(X),\ Z \in \operatorname{ran}(X),\ Z = \operatorname{ran}(X)$

(v) If ϕ is Δ_0, then so are $(\forall z \in \operatorname{dom}(X))\phi$ *and* $(\forall z \in \operatorname{ran}(X))\phi$

(vi) X *is a relation,* f *is a function,* $y = f(x)$

Proof.
(i)
$x = \{a, b\} \Leftrightarrow (\forall u \in x)(u = a \vee u = b)$
$x = (a, b) \Leftrightarrow (\forall u \in x)(u = \{a, b\} \vee u = a)$
$x = \emptyset \Leftrightarrow (\forall u \in x)u \neq u$
$x \subset y \Leftrightarrow (\forall u \in x)u \in y$
(ii)
x is transitive $\Leftrightarrow (\forall u \in x)u \subset x$
x is an ordinal $\Leftrightarrow x$ is transitive $\wedge\ (\forall u \in x)(\forall v \in x)(u \in v \vee v \in u \vee u = v)$
$(\forall u \in x)(\forall v \in x)(\forall w \in x)(u \in v \in w \Rightarrow u \in w)$

So we're saying (on the first line) x is transitive and the ordering is antisymmetric and (on the second line) the ordering is transitive; this gives the definition of an ordinal.
(iii)
$Z = X \times Y \Leftrightarrow (\forall u \in Z)(\exists a \in X)(\exists b \in Y)(u = (a, b))$
$Z = X - Y \Leftrightarrow (\forall u \in Z)(u \in X \wedge u \notin Y)$
$Z = X \cap Y \Leftrightarrow (\forall u \in Z)(u \in X \wedge u \in Y)$

$Z = \bigcup X \Leftrightarrow (\forall u \in Z)(\exists x \in X)(u \in x)$
(iv)
$Z \in \mathrm{dom}(X) \Leftrightarrow (\exists x \in X)(\exists u \in X)(\exists v \in u) x = (z, v)$
$Z = \mathrm{dom}(X) \Leftrightarrow (\forall z \in Z)\ z \in \mathrm{dom}(x) \wedge (\forall z \in \mathrm{dom}(x))\ z \in Z$
An assertion similar to the previous argument proves the statement for $\mathrm{ran}(X)$.
(v)
$(\forall z \in \mathrm{dom}(X))\phi \Leftrightarrow (\exists x \in X)(\exists u \in X)(\forall z, v \in u)(x = (z, v) \Rightarrow \phi)$
An assertion similar to the previous argument proves the statement for $\mathrm{ran}(X)$.
(vi)
X is a relation $\Leftrightarrow (\forall x \in X)(\exists u \in \mathrm{dom}(X))(\exists v \in \mathrm{ran} X) x = (u, v)$
f is a function $\Leftrightarrow$ f is a relation

$$\wedge (\forall x \in \mathrm{dom} f)(\forall y, z \in \mathrm{ran}(X))((x, y) \in f \wedge (x, z) \in f \Rightarrow y = z)$$

Where $(x, y) \in f \Leftrightarrow (\exists u \in f) u = (x, y)$, which is Δ_0, so the whole thing is Δ_0.
$y = f(x) \Leftrightarrow y \subset f(x) \wedge f(x) \subset y$

□

Here's a result that is easy to see, but very useful:

Lemma 3. *If ϕ is a formula, M is a transitive set, then ϕ^M is a Δ_0 formula.*

Proof. This is because all the $(\forall x)$ are replaced by $(\forall x \in M)$, i.e. the quantifiers are now bounded. (If we want to prove it explicitly, we can use induction on the complexities of ϕ). □

Remark. This only works if M is a *set*. It wouldn't work for a transitive class, which is often the case in this essay.

Lemma 4. *If M is a transitive class, and ϕ is a Δ_0 formula, then for all $x_1 ..., x_n \in M$,*

$$\phi^M(x_1 ... x_n) \Leftrightarrow \phi(x_1 ... x_n) \tag{*}$$

If (*) hold, we say that the formula ϕ is absolute for the transitive model M.

Proof. The lemma is easily verified for atomic formulae, and for formulae of the form $\psi \wedge \psi$, $\psi \vee \psi$, $\neg\phi$, $\phi \Rightarrow \psi$, $\phi \Leftrightarrow \psi$.

Now, suppose that ψ is Δ_0, let us verify (*) for $\phi = (\exists u \in x)\psi(u, x, ...)$. The proof for the $\forall$ case is similar.

$\Rightarrow$ Suppose ϕ^M holds, then we have $(\exists u \in M)(u \in x \wedge \psi^M)$. But by inductive hypothesis, we have $\psi^M \Leftrightarrow \psi$, so $(\exists u \in M)(u \in x \wedge \psi)$, i.e. $(\exists u \in x)\psi$.

$\Leftarrow$ If $(\exists u \in x)\psi$, then $x \in M$ and M is transitive means $u \in M$.

$\psi^M \Leftrightarrow \psi$ gives $\exists u(u \in M \wedge u \in x \wedge \psi^M)$ and so $((\exists u \in x)\psi)^M$. □

1.6 Lévy Hierarchy

A natural extension to the Δ_0 formulae is the Lévy Hierarchy. Introduced by Azriel Lévy, it is useful later when we are trying to examine and classify the definable formulae.

1.6.1 Definition

Δ_0 formulae are interesting, because all of its quantifiers are bounded. Now we try to generalise this by adding more quantifiers, either $\forall x$ or $\exists x$.

Definition 3 (Lévy Hierarchy). *(i) A formula is Σ_0 and Π_0 if its only quantifiers are bounded: i.e. Δ_0 formulae.*

(ii) A formula is Σ_{n+1} if it is the form $\exists x\ \phi$ where ϕ is Π_n.

(iii) A formula is Π_{n+1} if it is the form $\forall x\ \phi$ where ϕ is Σ_n.

(iv) A formula is Δ_n if it is both Σ_n and Π_n.

Remark.

1. The first point in definition may seem redundant but will make the notations cleaner.
2. We allow for dummy variables, so $\exists x \phi(u)$, where $\phi(u)$ is Δ_0, and x does not appear in $\phi(u)$, would be considered a Π_1 formula.
3. We can generalise this to other concepts by the obvious way:

   A property (class, relation) is Σ_n (or Π_n) if it can be expressed as a Σ_n (or Π_n) formula. A function F is Σ_n (or Π_n) if the relation $y = F(x)$ is Σ_n (or Π_n).

1.6.2 Properties

Lemma 5. *For $n \geq 1$:*

(i) If P, Q are Σ_n, then so are $\exists x P$, $P \wedge Q$, $P \vee Q, (\exists u \in x)P, (\forall u \in x)P$;

(ii) If P, Q are Π_n, then so are $\forall x P$, $P \wedge Q$, $P \vee Q, (\forall u \in x)P, (\exists u \in x)P$;

(iii) If P is Σ_n then $\neg P$ is Π_n; If P is Π_n then $\neg P$ is Σ_n

Proof. We prove $n = 1$. Induction will take care of the rest. Indeed, we only need $n = 1$ for the purpose of this article.
(i) Suppose

$$P(x) \Leftrightarrow \exists z\ \phi(z, x, ...)$$
$$Q(x) \Leftrightarrow \exists z\ \psi(z, x, ...)$$

Where ϕ, ψ are Δ_0 formulas. We have

$$\begin{aligned} \exists x P(x, ...) &\Leftrightarrow \exists x\ \exists z\ \phi(z, x, ...) \\ &\Leftrightarrow \exists v\ \exists w \in v\ \exists x \in w\ \exists z \in w\ (v = (x, z) \wedge \phi(z, x, ...)) \end{aligned} \tag{4}$$

Thus it is Σ_1. Further, we have

$$P(x, ...) \wedge Q(x, ...) \Leftrightarrow \exists z \exists u\ \phi(z, x, ...) \wedge \psi(u, x, ...)$$

$$P(x, ...) \vee Q(x, ...) \Leftrightarrow \exists z \exists u\ \phi(z, x, ...) \vee \psi(u, x, ...)$$

$$(\exists u \in x) P(u, ...) \Leftrightarrow \exists x \exists u\ u \in x \wedge \phi(z, u, ...)$$

These are Δ_1 because of what we just proved about $\exists x\ P$.

Now, $(\forall u \in x)P$ is a bit more complicated. $\exists x$ is in the wrong place, and we have to be careful: We cannot just move them around at will.

One way we can get around this is to see if we can bound x. If there is a certain y, such that for all $u \in x$, $\exists z \in y$ such that $\phi(z, u, ...)$ holds, then we have the following:

$$\begin{aligned} (\forall u \in x)\ P(u, ...) &\Leftrightarrow (\forall u \in x)\ \exists z\ \phi(z, u, ...) \\ &\Leftrightarrow (\exists y)\ (\forall u \in x)\ (\exists z \in y)\ \phi(z, u, ...) \end{aligned} \tag{5}$$

Which would solve the problem, as the equation is now Σ_1.

But we notice that this is a very familiar problem: given x such that $\phi(x)$ holds, find y such that $(\exists x \in y)$ such that $\phi(x)$ holds. Indeed, the Collection Principle solves this problem perfectly.

Let us remember the content of the Collection Principle:

$$(\forall x)(\exists y)(\forall u \in x)[(\exists z)\phi(u, z, p...) \Rightarrow (\exists z \in y)\phi(u, z, p...)]$$

Once we have the Collection Principle, we are done, as it gives the last equivalence in 5. The new sentence on the right hand side of that equivalence is Σ_1.

(iii)

$$\neg \exists z\ \phi(z, x, ...) \Leftrightarrow \forall z\ \neg \phi(z, x, ...)$$

$$\neg \forall z\ \phi(z, x, ...) \Leftrightarrow \exists z\ \neg \phi(z, x, ...)$$

(ii) follows from (i) and (iii). □

Lemma 6. Δ_1 *properties are absolute for transitive models.*

Proof. Δ_0 properties are absolute for all transitive models.

Σ_1 **properties are *downward absolute*:**

If $P(x) = (\exists z)\phi(x, z, ...)$ where ϕ is Δ_0, then

$$P^M(x) = (\exists z \in M)\phi^M(x, z, ...) \Leftrightarrow (\exists z \in M)\phi(x, z, ...)$$

Thus if M is a transitive model, then for all $x \in M$,

$$P^M(x) \Rightarrow P(x)$$

Π_1 **properties are *upward absolute*:**
If $P(x) = (\forall z)\phi(x, z, ...)$ where ϕ is Δ_0, then

$$P^M(x) = (\forall z \in M)\phi^M(x, z, ...) \Leftrightarrow (\forall z \in M)\phi(x, z, ...)$$

Thus if M is a transitive model, then for all $x \in M$

$$P(x) \Rightarrow P^M(x)$$

Thus Δ_1 properties are absolute for transitive models. $\square$

1.7 Elementary Submodels

When we are trying to find (or at least bound) the cardinality of something, one very powerful tool is Mostowski collapse, as it gives us a isomorphism onto a transitive set, which is hopefully easier to study.

Suppose we have two sets M and N, and we want to know if we can collapse M down to N. This is probably not possible, but if it were, the sets had better be 'similar' in some sense.

Indeed, Mostowski collapse gives us an isomorphism, so a good thing to guess is that if M and N satisfy the same formulae, then perhaps M can be collapsed down to N?

There is a name for such 'similar' models:

Definition 4. *Let $\mathcal{A} = (A, P, ...F, ...c, ...)$ be a model, where P are the relations, F are the functions, c are the constants etc.*

1. *Submodel of A is a subset $B \subset A$ endowed with the relations $P^{\mathcal{A}} \cap B^n$, $F^{\mathcal{A}} \cap B^n$, $c^{\mathcal{A}}$, where $c^{\mathcal{A}} \in B$ etc.*

2. *A submodel $\mathcal{B} \subset \mathcal{A}$ is an elementary submodel $\mathcal{B} \prec \mathcal{A}$ if for every formula ϕ, and every $a_1, ..., a_n \in B$,*

$$\mathcal{B} \models \phi[a_1, ..., a_n] \text{ iff } \mathcal{A} \models \phi[a_1, ..., a_n]$$

3. *Two models $\mathcal{A}, \mathcal{B}$ are elementarly equivalent if they satisfy the same sentences.*

1.7.1 Skolem Functions

How can we construct elementary submodels? The following lemma gives us some ideas:

Lemma 7. *A subset $B \subset A$ forms an elementary submodel of $\mathcal{A}$ iff for every formula $\phi(u, x_1, ...x_n)$, and every $a_1, ...a_n \in B$:*

$$\text{If } \exists a \in A \text{ s.t. } \mathcal{A} \models \phi[a, a_1, ...a_n] \text{ then } \exists a \in B \text{ s.t. } \mathcal{A} \models \phi[a, a_1, ...a_n] \quad (6)$$

Proof. Consider the formula $\psi = \exists a\ \phi(a, a_1, ...a_n)$.

We have $\mathcal{A} \models \psi$, thus $B \models \psi$ because B is an elementary submodel, so this says $\exists a \in B$ s.t. $\mathcal{A} \models \phi[a, a_1, ...a_n]$. □

Now we'll try to construct submodels which are 'simpler' or 'smaller'.

We are reminded of the Downward Löwenheim–Skolem theorem, which says if we have a theory in a countable language, and the theory has a model, then it has a countable model. This seems to be along the same lines of what we are trying to do. Turns out there is a generalisation of this theorem, using Skolem function, which we will outline now.

Lemma 8 (Generalisation of Löwenheim–Skolem)**.** *Suppose we have a model $\mathcal{A}$ in language $\mathcal{L}$, and a set $X \subset A$, then there exist an elementary submodel of $\mathcal{A}$ containing X, with cardinality at most $|X| \cdot |\mathcal{L}| \cdot \aleph_0$.*

Remark.

1. Downward Löwenheim–Skolem theorem for countable language $\mathcal{L}$ follows easily from this lemma.

2. For this proof, we will assume the Axiom of Choice.

Proof. Let us first fix ϕ, then we can define, a function $h : A^n \to A$, which takes $a_1, ..., a_n$ as input, and output b.

This is called a *Skolem function* for ϕ, and it satisfies:

$$(\exists a \in A)\mathcal{A} \models \phi(a, a_1, ..., a_n) \quad \text{implies} \quad \mathcal{A} \models \phi[h(a_1, ..., a_n), a_1, ..., a_n]$$

Using Axiom of Choice, we can construct a function for every ϕ.

If a subset $B \subset A$ is closed under the Skolem functions, for all formulas, then B satisfies the Equation 6, and hence form an elementary submodel of $\mathcal{A}$.

Given a set of Skolem functions, one for each formula of $\mathcal{L}$, the closure of a set $X \subset A$ is a *Skolem Hull* of X. It is clear that the Skolem hull of X is an elementary submodel of $\mathcal{A}$ and contains X.

Now, let's look at the cardinality: Given a language $\mathcal{L}$, the formulae of length n, has cardinality $|\mathcal{L}|^n = |\mathcal{L}|$. As all the formulae are of finite length, there are at most $|\mathcal{L}| \cdot \aleph_0$ formulae.

Now, fix a formula ϕ: it has a finite number of variables, say it has n variables, then let us look at the image of X^n under ϕ: i.e. look at all possible outputs of ϕ. All of these outputs are going into the convex hull. The cardinality of the outputs is at most $|X|^n = |X|$. This holds for every formula, so the convex hull has cardinality at most $|X| \cdot |\mathcal{L}| \cdot \aleph_0$. □

2 Constructible Sets

2.1 Definition

Fix a model $(M, \in)$. In this model, we have formulae made of finite list of symbols, consisting of objects in M and the relationship operator $\in$. Let *Form* be the list of such formulae in M.

Definition 5 (Definable Sets)**.** *We say that a set X is* definable *if there exist a formula $\phi \in Form$ and $a_1...a_n \in M$ such that $X = \{x \in M : (M,E) \models \phi(x, a_1, ...a_n)\}$.*

Let def*$(M) = \{X \subset M : X$ is definable over $(M,E)\}$*

Definition 6. *We define by transfinite induction:*

(i) $L_0 = \emptyset,\ L_{\alpha+1} = def(L_\alpha)$

(ii) $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ *for a limit ordinal* λ

(iii) $L = \bigcup_{\alpha\in Ord} L_\alpha$

L is the class of of constructible sets.

2.2 Properties

Let us develop some intuition for L by proving the following lemmas:

Lemma 9. *$\forall\alpha$, L_α is a transitive set.*

Proof. [2] Looking at the definition for constructible sets, we have that $M \in$ def(M), because we can let ϕ be the formula $x = x$, and $\forall a \in M$, $a \in$ def(M), by letting ϕ be $x = a$. Further, def(M) is a subset of M, thus $M \subset$ def(M) $\subset \mathcal{P}(M)$.

In particular, this means that $L_{\alpha+1}$ is transitive:

$\forall\alpha$, $L_\alpha \subset L_{\alpha+1} \subset \mathcal{P}(L_\alpha)$, so $(\forall x)(\forall y \in L_{\alpha+1})\ x \in y \Rightarrow x \in L_\alpha \Rightarrow x \in L_{\alpha+1}$

And if α is a limit ordinal, it is easy to check that L_α is transitive. □

Lemma 10. *Let α be an ordinal. Then $\alpha \subset L_\alpha$, and $L \cap Ord = \alpha$*

Proof. It is natural to prove this by induction, as we are dealing with ordinals and an inductive definition.

Because $L_{\alpha+1} \subset \mathcal{P}(L_\alpha)$, if we can show $\alpha \in L_{\alpha+1}$, then $\alpha \subset L_\alpha$. i.e. we show that α is a definable subset of L_α.

$\alpha = \{x \in L_\alpha : x$ is an ordinal $\}$. This is almost good, except 'x is an ordinal' is a statement in V. This is a definable set over V, and we want a set which is definable over L_α.

However, we note that 'x is an ordinal' is a Δ_0 formula, so in the transitive class L_α, x is an ordinal $\iff L_\alpha \models x$ is an ordinal.

This solves the problem, because now $\alpha = \{x \in L_\alpha : L_\alpha \models x$ is an ordinal $\}$, so it is in $L_{\alpha+1}$. □

[2]Nothing in this proof is particular to definable sets. In fact, as long as W_α is a *cumulative hierarchy* of sets:

(i) $W_0 = \emptyset$

(ii) $W_\alpha \subset W_{\alpha+1} \subset \mathcal{P}(W_\alpha)$

(iii) if α is limit ordinal then $W_\alpha = \bigcup_{\beta<\alpha} W_\beta$

Then each W_α is transitive, and indeed $W_\alpha \subset V_\alpha$.

3 L is a model of ZF

Since L is transitive, every Δ_0 formula is absolute for L. This means that suppose we have a set $Y = \{x \in L_\alpha : \phi(x)\}$, we have $Y = \{x \in L_\alpha : L_\alpha \models \phi(x)\}$, i.e. Y is definable over L_α. We have seen this argument once already in Lemma 10, and we will use this extensively when checking the formulae of ZF:

Extensionality. L is transitive, thus extensional.

Pair Set. Given $a, b \in L$, let $c = \{a, b\}$. We wish to show $c \in L$. Suppose $a, b \in L_\alpha$, then $\{a, b\}$ is definable simply by the formula '$x = \{a, b\} \Leftrightarrow u \in x \wedge v \in x \wedge (\forall w \in x)(w = u \vee w = v)$'. We note that this is a Δ_0 formula, so using this formula, Y is definable over L_α.

Union. Given $X \in L$, let $Y = \bigcup X$, and want to show $Y \in L$. Let α be such that $X \in L_\alpha$, and $Y \subset L_\alpha$. Y is definable over V by the formula '$x \in Z \wedge Z = \bigcup X$'. Since $Z = \bigcup X \Leftrightarrow (\forall z \in Z)(\exists x \in X) z \in x \wedge (\forall x \in X)(\forall z \in x) z \in Z$, it is Δ_0. Thus Y is defined by a Δ_0 formula, so it is definable over L_α.

Power set. Given $X \in L$, let $Y = \mathcal{P}(X) \cap L$. The intuition is that this is certainly the power set of X restricted to L, but there are two things we have to be careful about: First of all, Y is a set in V but not necessarily in L, and secondly, L could technically have a different structure, and the power set inside it may not be the intersection. We'll have to check these points.

Let α be such that $Y \subset L_\alpha$. Y is definable over L_α by formula $x \subset X$. Now, $x \subset X \Leftrightarrow (\forall u \in x) u \in X$, which is Δ_0, so $Y \in L$.

Next, we'll show that $Y = \mathcal{P}^L(X)$, i.e. 'Y is power set of X' holds in L. But $x \in Y \Leftrightarrow x \subset X$ is also Δ_0, so this means $L \models x \in Y \Leftrightarrow x \subset X$. This is true for every $x \in L$, so $Y = \mathcal{P}^L(X)$.

Foundation. If $S \in L$ is nonempty, let $x \in S$ be such that $x \cap S = \emptyset$. Intuitively, this should be the 'smallest set' we're looking for.

Firstly, $x \in L$ as L is transitive.

Now we'll show that '$x \cap S = \emptyset$' holds in L. Note that '$x \cap S = \emptyset \Leftrightarrow Y = \emptyset \wedge Y = x \cap S$', but we $Y = x \cap S \Leftrightarrow (\forall y \in Y) y \in x \wedge y \in S$, so this formula is Δ_0, so done.

Empty Set. $\emptyset \in L$. $(\forall y \in L) y \notin \emptyset$, so $L \models (\exists S)(\forall y)(y \notin S)$

i.e.: '$\emptyset \Rightarrow \emptyset^L$'. Indeed, as L is transitive, we actually have '$\emptyset^L \Rightarrow \emptyset$' too: if S is empty in L, it is empty in V.

Infinity. $\omega \in L$, so the intuition is that this is an infinite set. But we'll have to show L also sees ω as an infinite set.

We wish to show that $L \models \exists S\ (\emptyset \in S \wedge (\forall x \in S) x \cup \{x\} \in S)$

In the above formula, the notions involved are pair set ($\{x\} = \{x, x\}$), empty set and union. But we have already shown that

$$\{a,b\}^L = \{a,b\}, \qquad \cup^L X = \cup X, \qquad \emptyset^L = \emptyset$$

Thus ω satisfies that formula in L.

Separation. Take a formula ϕ, we wish to show that $Y = \{u \in X : \phi^L(u,p)\}$ is in L.

We notice that Y is a definable set over L, which is almost good. Also, we know that there is a α such that $X, p \in L_\alpha$. At the moment this Y is in L, can we pass it on to L_α?

But this is exactly what the Reflection Principle is for: it gives us α such that $X, p \in L_\alpha$ and $Y = \{u \in X : \phi^{L_\alpha}(u,p)\}$. Thus $Y = \{u \in L_\alpha : L_\alpha \models u \in X \wedge \phi(u,p)\}$. Thus $Y \in L$.

Replacement.

Take a function class F in L. We remind ourselves that a function class is made up of ordered pairs, so we wish to show all of those pairs are in L.

$\forall X \in L$, $F(X)$ is a set in V. However, because F is in L, $F(X) \subset L$, and it is a set so strictly smaller than L, and $\exists \alpha : F(X) \subset L_\alpha$.

Now we proof that this is indeed the image of X in L, under F using Separation.

Suppose that p is the formula that defines F, i.e.

$$(\forall x)(\forall y)(\forall z)((p \wedge p[z/y]) \Rightarrow y = z), \quad (x,y) \in F \Leftrightarrow p(x,y)$$

Then, separation (in L) gives:

$$L \models (\exists Y)(y \in Y \Leftrightarrow y \in L_\alpha \wedge (\exists x \in X)p(x,y))$$

And $y \in L_\alpha \wedge (\exists x \in X)p^L(x,y) \Leftrightarrow y \in F(X)$, so $Y = F(X)$ is the image of X under F in L.

4 Gödel Normal Form: A Description of def(M)

If we wish to describe a set in def(M), we have to say $\exists\phi$... but the problem is, there is no systematic way of listing ϕ, thus there is no concrete way of describing def(M). In this section, we try to make def(M) more tangible.

We have constructed all the Δ_0 formulae inductively, with some basic operations such as $\phi \cap \psi$, $\neg\phi$ etc. Can we describe the construction of a set with some elementary operations too?

Let us guess some candidates. Some basic set-building operations had better be there: union, intersection, pair set.

The others are less obvious, but Gödel found ten operations (in some books eight) which completely defines def(M).

Definition 7. *(Gödel Operations)*

$$
\begin{aligned}
&G_1(X,Y) = \{X,Y\},\\
&G_2(X,Y) = X \times Y,\\
&G_3(X,Y) = \epsilon(X,Y) = \{(u,v) : u \in X \wedge v \in Y \wedge u \in v\},\\
&G_4(X,Y) = X - Y,\\
&G_5(X,Y) = X \cap Y,\\
&G_6(X) = \bigcup X,\\
&G_7(X) = \mathrm{dom}(X),\\
&G_8(X) = \{(u,v) : (v,u) \in X\},\\
&G_9(X) = \{(u,v,w) : (u,w,v) \in X\},\\
&G_{10}(X) = \{(u,v,w) : (v,w,u) \in X\}.
\end{aligned} \tag{7}
$$

Remark. Note that these operations are NOT set theoretic formulae, they are operations, acting on sets.

Theorem 3 (Gödel's Normal Form Theorem)**.** *If $\phi(u_1,...u_n)$ is a Δ_0 formula, then there is a composition G of $G_1,...G_{10}$ such that for all $X_1,...,X_n$,*

$$G(X_1,...,X_n) = \{(u_1,...,u_n) : u_1 \in X_1,...u_n \in X_n \text{ and } \phi(u_1,...,u_n)\}. \tag{8}$$

Proof. There is going to be case work involved: let's see if we can list out all the possible forms of ϕ first.

If we have '$x = y$', we can replace this with '$(\forall u \in x)u \in y \wedge (\forall v \in y)v \in x$'; $x \in x$ can be replaced by $(\exists u \in x)u = x$.

Another thing: We do not want quantifiers of the form $(\exists x_1 \in x_2)(\exists x_3 \in x_1)$. We do not want to do the induction only to discover later that the there are additional requirements on x_1. Thus, let us rename the bounded variables in $\phi(u_1,...u_n)$ such that the variable with the highest index is quantified first.

One last thing: We allow dummy variables, so $\phi(x_1,...x_n) = (x_1 \in x_2)$ and $\phi(x_1,...x_{n+1}) = (x_1 \in x_2)$ are different formulae.

Thus: all Δ_0 formulae can be written in one of the following forms:

(i) The only logical symbols in ϕ are $\neg$, $\wedge$, and restricted $\exists$;

(ii) $=$ does not occur;

(iii) The only occurrence of $\in$ is $u_i \in u_j$, where $i \neq j$;

(iv) ϕ is of the form $\exists(u_{m+1} \in u_i)\psi(u_1,...,u_{m+1})$ where $i \leq m$.

Now, we prove the theorem case by case. To make reading easier, we will omit some cases where the proof is repetitive. The full proof can be found in the Appedix.

Case I. $\phi(u_1, ...u_n)$ is an atomic formula $u_i \in u_j (i \neq j)$. We prove this case by induction on n. The cases for $n > 2$ are similar in spirit, so we will omit it here and only prove it for $n = 2$.

Here we have

$$\{(u_1, u_2) : u_1 \in X_1 \wedge u_2 \in X_2 \wedge u_1 \in u_2\} = \epsilon(X_1, X_2)$$

and

$$\{(u_1, u_2) : u_1 \in X_1 \wedge u_2 \in X_2 \wedge u_2 \in u_1\} = G_8(\epsilon(X_2, X_1)).$$

Case II. $\phi(u_1, ..., u_n)$ is a negation, $\neg\psi(u_1, ...u_n)$. By the induction hypothesis, there is a G such that:

$$\{u_1, ..., u_n : u_1 \in X_1, ...u_n \in X_n \text{ and } \psi(u_1, ...u_n)\} = G(X_1, ...X_n).$$

The set we want is easily achieved via G_4:

$$\begin{aligned}\{u_1, ..., u_n : u_1 \in X_1, ...u_n \in X_n \text{ and } \neg\psi(u_1, ...u_n)\}& \\ = X_1 \times ... \times X_n - G(X_1, ...X_n).&\end{aligned}$$

Case III.$\phi = \psi_1 \wedge \psi_2$. This case is very easy to handle, using the induction hypothesis, and G_5 intersection.

Case IV. $\phi(u_1, ...u_n)$ is the formula $(\exists u_{n+1} \in u_i)\psi(u_1, ...u_{n+1})$.

Let $\chi(u_1, ...u_{n+1})$ be the formula $\psi(u_1, ...u_{n+1}) \wedge u_{n+1} \in u_i$. By induction hypothesis (χ is less complex than ϕ, because there is one less quantifier), there is a G such that

$$\begin{aligned}\{(u_1, ...u_{n+1}) : u_1 \in X_1, ..., u_{n+1} \in X_{n+1} \text{ and } \chi(u_1, ...u_{n+1})\}& \\ = G(X_1, ...X_{n+1})&\end{aligned}$$

For all $X_1, ...X_{n+1}$. We claim that

$$\begin{aligned}\{(u_1, ...u_n) : u_1 \in X_1, ...u_n \in X_n \text{ and } \phi(u_1, ...u_n)\}& \\ = (X_1 \times ... \times X_n) \cap \operatorname{dom}(G(X_1, ...X_n, \bigcup X_i))&\end{aligned} \tag{9}$$

Let us denote $u = (u_1, ..., u_n)$ and $X = X_1, ...X_n$. For all $u \in X$, we have

$$\begin{aligned}\phi(u) &\Leftrightarrow (\exists v \in u_i)\psi(u_i, v) \\ &\Leftrightarrow \exists v(v \in u_i \wedge \psi(u, v) \wedge v \in \bigcup X_i) \\ &\Leftrightarrow u \in \operatorname{dom}\{(u, v) \in X \times \bigcup X_i : \chi(u, v)\}\end{aligned}$$

And (9) follows, and this completes the proof. □

Now we have shown that a set described by Δ_0 formula is describable by a composition of G_i, now we prove kind of the converse of this:

Lemma 11. *If G is a Gödel operation, then the property $Z = G(X_1, ...X_n)$ can be written as a Δ_0 formula.*

Remark.

1. This shows that the property $Z = G(X_1, ...X_n)$ is absolute for the transitive models.

2. This statement is plausible, because this holds for several G_i's, for example $Z = \{X, Y\}$. Refer to Lemma 2 to see these are Δ_0 formulae.

Proof.
Let us start by checking the lemma for all G_i. Lemma 2 takes care of all except for G_3, G_8, G_9, G_{10}.

$$\begin{aligned} &Z = G_8(X) \\ &\quad \Leftrightarrow (\forall z \in Z)(\exists x \in X)(\exists u \in \mathrm{ran}(X))(\exists v \in \mathrm{dom}(X))(x = (v, u) \wedge z = (u, v)) \\ &\quad \wedge (\forall x \in X)(\forall u \in \mathrm{ran}(X))(\forall v \in \mathrm{dom}(X)(\exists z \in Z)(x = (v, u) \Rightarrow z = (u, v))) \end{aligned}$$

This is Δ_0. This looks all very complicated, but it is just a rephrasing of $Z = G_8(X)$.

We can do a similar rephrasing for G_3, G_9, G_{10}, the details we will omit here.

The general case is not easy to see. First we notice that

$$Z = G(X, ...) \Leftrightarrow (\forall u \in Z)\ u \in G(X, ...) \wedge \forall u \in G(X, ...)\ u \in Z.$$

So we will try to show that $u \in G(X, ...)$ and $(\forall u \in G(X, ...))\ u \in Z$ are Δ_0 functions.

To prove this, we use induction on complexity again because we have already prove the lemma for $G = G_i$. In fact, we will have to have a induction hypothesis involving four parts, and prove all parts simultaneously.

Claim.

(i) $u \in G(X, ...)$ is Δ_0.

(ii) If ϕ is Δ_0, then so are $\forall u \in G(X, ...)\phi$ and $\exists u \in G(X, ...)\phi$.

(iii) $Z = G(X, ...)$ is Δ_0.

(iv) If ϕ is Δ_0, then so is $\phi(G(X, ...))$.

Proof. Step 1: Check the claims for all G_i. We will not check all four cases and all i, the ideas are similar and involves simple reformulations. Much of the techniques are already displayed in Lemma 2. For example, to show that $u \in G_1(X, ...)$ is Δ_0, observe $u \in \{x, y\} \Leftrightarrow u = x \vee x = y$, which is Δ_0. We will leave the rest to the reader.

Step 2: Now, let us build up all the compositions of Gaussian operators by induction, proving (i) to (iv). We will again show this for some typical examples and leave the rest to the reader. Remember that the induction hypothesis has four parts.

(i): We will prove this for two examples, $\{F, G\}$, and $F \times G$.

Suppose $x = F(X, ...)$ and $x = G(X, ...)$ are Δ_0 formulae, we have $u \in \{F(X, ...), G(X, ...)\} \Leftrightarrow u = F(X, ...) \vee x = G(X, ...)$. From the induction hypothesis, $u = F(X, ...)$ and $x = G(X, ...)$ are Δ_0, thus this is also Δ_0.

Similarly, $u \in F(X, ...) \times G(X, ...)$ can be written as the following:

$$\exists x \in F(X, ...)\ \exists y \in G(X, ...) u = (x, y).$$

Now, because F, G are less complex than $F \times G$, we can assume by induction hypothesis (ii) that $\exists y \in G(X, ...) u = (x, y)$ is Δ_0, and as a consequence $\exists x \in F(X, ...) \exists y \in G(X, ...) u = (x, y)$ is also Δ_0.

(ii): We will prove this for the example of $\{F, G\}$. Consider the formula $\forall u \in \{F(X, ...), G(X, ...)\} \phi(u)$. This can be written as

$$\phi(F(X, ...)) \wedge \phi(G(X, ...))$$

Then from induction hypothesis (iv), we are done.

(iii): The proof follows from (i) and (ii), because $Z = G(X, ...) \Leftrightarrow (\forall u \in Z)\ u \in G(X, ...) \wedge \forall u \in G(X, ...)\ u \in Z$.

(iv): Let ϕ be a Δ_0 formula, then consider our language and how formulae are built up, we see that $G(X, ...)$ occurs in $\phi(G(X, ...))$ in the form $u \in G(X, ...)$, $G(X, ...) \in u$, $Z = G(X, ...)$, $\forall u \in G(X, ...)$ or $\exists u \in G(X, ...)$. Since $G(X, ...) \in u$ can be replaced with $(\exists v \in u) v = G(X, ...)$, we use (i)-(iii) to show that $\phi(G(X, ...))$ is a Δ_0 property. □

Once we have this claim, we can finish the induction. Remember:

$$Z = G(X, ...) \Leftrightarrow (\forall u \in Z) u \in G(X, ...) \wedge \forall u \in G(X, ...) u \in Z.$$

Thus the claim proves that $Z = G(X, ...)$ is Δ_0. □

Once we have these lemmas, we arrive at a very concrete way of describing $def(M)$:

Lemma 12. *For every transitive set M,*

$$\mathrm{def}(M) = \mathrm{cl}(M \cup \{M\}) \cap \mathcal{P}(M)$$

Proof. If ϕ is a formula, then ϕ^M is a Δ_0 formula. Thus, by the Gödel Normal Form Theorem, there exist a Gödel operation G such that for every transitive set M and all $a_1, ...a_n$,

$$\begin{aligned}\{x \in M : M \models \phi[x, a_1, ...a_n]\} &= \{x \in M : \phi^M(x, a_1, ...a_n)\} \\ &= G(M, a_1, ...a_n)\end{aligned}$$

Thus the set $\{x \in M : M \models \phi[x, a_1, ...a_n]\}$ is in the closure of $M \cup \{M\}$ under $G_1, ...G_{10}$: i.e. take elements of $M \cup \{M\}$, perform operations $G_1, ..., G_{10}$ on it and it stays in $\text{cl}(M \cup \{M\})$. Note $\{M\}$ is there, because our operators can take in M as an argument.

Conversely: If G is a composition of $G_1, ...G_{10}$ then by Lemma 11 we have Δ_0 formula ϕ such that for all M and all $a_1, ...a_n$, if $X \in \text{cl}(M \cup \{M\})$, i.e. $X = G(M, a_1, ...a_n)$ for some G, then $X = \{x : \phi(M, x, a_1, ...a_n)\}$.

This is almost the perfect converse, except here ϕ lives in V, and we are looking for something defined by $\phi \in$ *Form*, which lives in M.

In more detail: ϕ is a set theoretic, Δ_0 formula, and all of its quantifiers are bounded. we want some formula in *Form*, which lives inside of M, whose quantifiers are no longer bounded because it only sees the elements in M anyway.

But the situation is easy to fix because $X \subset M$, so we can just modify ϕ into something living in *Form* by replacing each bounded quantifier $\exists u \in M$ by $\exists u$. We will refer to this new formula in *Form* by the name ψ. Then we have $X = \{x \in M : M \models \psi[x, a_1, ...x_n]\}$.

□

5 Axiom of Constructibility

Our goal in this section is to show that L satisfies $V = L$: i.e. we want to show that $\forall x \in L$, ($L \models x$ is constructible). However, by the definition of L, $x \in L \Rightarrow$ (x is constructible (in V)). Thus what we would like to show it 'x is constructible' is absolute:

$$(\text{x is constructible})^{\text{L}} \Leftrightarrow (\text{x is constructible}) \tag{10}$$

5.1 Absoluteness of Constructibility

Because L contains all ordinals, if x is constructible, it must be contained in L_α for some α (but be careful to work in L):

$$(\text{x is constructible})^{\text{L}} \Leftrightarrow \exists \alpha \in L \; x \in L_\alpha^L \tag{11}$$

If we can show that $x \in L_\alpha \Leftrightarrow x \in L_\alpha^L$, somehow, then we're done. Thus what we want to show is that the function $\alpha \mapsto L_\alpha$ is absolute.

But to work with L_α, transfinite induction is inevitable, and we have no idea how to work with that yet. Thus we have the following lemma:

Lemma 13. *Let $n \geq 1$, let G be a Σ_n function (on V), and let F be defined by induction:*

$$F(\alpha) = G(F|_\alpha)$$

Then F is a Σ_n function on Ord.

Proof. Let us start by noticing that 'x is an ordinal' is a Δ_0 formula. So it is enough to verify that the following expression is Σ_n, because that's just how F is defined:

$$y = F(\alpha) \iff \exists f(f \text{ is a function} \wedge \mathrm{dom}(f) \\ = \alpha \wedge (\forall \xi < \alpha) f(\xi) = G(f|_\xi) \wedge y = G(f) \tag{12}$$

All properties and operations in this equation is Σ_0, and G is Σ_n.

Now, if $\phi(x, p...)$ is a Δ_0 formula which does not have u as a variable, then $\phi(x, p...) \Leftrightarrow (\exists u)\phi(x, p, ...)$, hence it can also be considered as a Σ_n expression; Combine this with what we know about $P \wedge Q$, we get $y = F(\alpha)$ is Σ_n. □

Once we have this, let's move on to the Lemma we want to prove:

Lemma 14. *The function $\alpha \mapsto L_\alpha$ is Δ_1, which we know is absolute.*

Remark. Look at the right hand side in Equation 11:

'$(\exists\alpha)(x \in L_\alpha)$' is different in nature to things we've seen like '$(\exists x)(x \in y)$'. Here, we are saying for the function $F : \alpha \mapsto L_\alpha$, $(\exists\alpha)(x \in F(\alpha))$, and that's why here we need to show F is absolute.

Proof. By Lemma 13, we just need to show that the induction step is Δ_1.

The induction step consists of taking unions (which is Δ_0) and $L_{\alpha+1} = \mathrm{def}(L_\alpha) = \mathrm{cl}(L_\alpha \cup \{L_\alpha\} \cap \mathcal{P}(L_\alpha))$.

First of all, look at the operation Def. We have that

$$(\forall x)(x \in \mathrm{Def}(A) \iff x \text{ is definable over } A \text{ with parameters from } A).$$

On the right hand side, all the parameters are bounded, thus the expression '$(x \in \mathrm{Def}(A) \iff x$ is definable over A with parameters from A)' is Δ_0. Thus we see that Def(A) is a Π_1 expression.

If we can prove that for any M, $Y = \mathrm{cl}(M)$ is Σ_1, we are done. But the expression '$Y = \mathrm{cl}(M)$' is equivalent to the following:

$$\exists W[W \text{ is a function} \wedge \mathrm{dom}(W) = \omega \wedge Y = \bigcup \mathrm{ran}(W) \wedge W(0) = M \\ \wedge (\forall n \in \mathrm{dom}(W))(W(n+1) = W(n) \cup \{G_i(x, y) : x, y \in W(n), i = 1...10\})]$$

In the previous equation, W is the thing that build the closure inductively: once we have $W(n)$, $W(n+1)$ is $W(n)$ together with the actions of G_i's on any pair of x, y.

$y = \mathrm{dom}(x)$, $y = \mathrm{ran}(x)$, 'x is a function', union are all Δ_0, so all the stuff in the square bracket is Δ_0 ($\{G_i(x), y\}$ can be written as two unions over $W(n)$, so Δ_0). Thus the whole equation is Σ_1, which is what we want. □

Now that we've shown this, it is easy to finish up:

Corollary 2. *The property 'x is constructible' is absolute for inner models of ZF.*

Proof. (x is constructible)^M $\Leftrightarrow$ $(\exists\alpha \in M)x \in L^M_\alpha$ $\Leftrightarrow$ $(\exists\alpha)$ $x \in L_\alpha$ $\Leftrightarrow$ x is constructible. □

Now we can prove the Axiom of Constructibility:

Theorem 4 (Gödel)**.** *L satisfies the Axiom of Constructibility*

Proof. For every $x \in L$, '(x is constructible)^L' iff x is constructible, hence 'every set is constructible' holds in L. □

Before leaving this section, let us notice another important corollary that arises from the proof of 14:

Corollary 3. *The operation def is absolute.*

Proof. Def is Π_1, closure is Σ_1. Thus $\mathrm{Def}(M) = \mathrm{cl}(M \cup \{M\}) \cap \mathcal{P}(M)$ is absolute. □

5.2 Inner Models of ZF

We have shown that L is a model of ZF, and that L lives in the universe V. So it is natural to think of L as an *inner model*.

Definition 8. *An inner model is a transitive class that contains all ordinals, and satisfy the axioms of ZF.*

Thus L is an inner model. In this section we will now show that it is the smallest inner model.

Theorem 5. *L is the smallest inner model of ZF*

Proof. If M is an inner model, then L^M (the class of all constructible sets in M) is L. This is because absoluteness of constructibility holds for all inner models. So $L \subset M$. □

6 Consistency of Generalised Continuum Hypothesis

Now we will explore GCH, which will then give us a well ordering of L and the Axiom of Choice. Much like in the previous section, we will show that for L, the general continuum hypothesis holds, and this shows that the axioms of ZF is consistent with GCH, which will be a very significant result.

6.1 Axiom of Choice

We will start off with the following lemma:

Lemma 15. *For all $\alpha \geq \omega$, $|L_\alpha| = |\alpha|$ in L.*

Remark. This result is also true in V, but it is important to remember to work in L here.

Proof. To start, $|L_\omega| = \omega$ is easy to see: L_n is finite for all natural number n. L_ω is a countable unions of countable sets, thus countable.

Now we proceed by transfinite induction. Suppose that $|L^L_\alpha| = |\alpha|$, then: $L^L_{\alpha+1} = \mathrm{Def}^L(L_\alpha)$. But Def is absolute, as we have just shown, thus we have

$$(x \in \mathrm{Def}(L_\alpha))^L \Leftrightarrow x \in L \cap \mathrm{Def}(L_\alpha) \Leftrightarrow x \in L \cap L_{\alpha+1} \Leftrightarrow x \in L_{\alpha+1}.$$

All elements of $L_{\alpha+1}$ is defined by some formula, with finite length, and constants over L_α, thus the number of such elements are less than or equal to $\omega \cdot |L_\alpha| = \omega \cdot |L_\alpha| = |L_\alpha|$. So this means $|L^L_{\alpha+1}| = |L_{\alpha+1}| = |\alpha| = |\alpha + 1|$.

And for limit ordinal λ, we have $|L_\lambda| > |\alpha|$ for all $\alpha < \lambda$, and $|L_\lambda| \leq \Sigma_{\alpha<\lambda}|\alpha|$. So $|L_\lambda| = \lambda$. □

Corollary 4. *L can be well-ordered.*

Proof. Each L_α bijects with ordinal α, so it is easy to see that L can be well ordered. □

Theorem 6. *Axiom of choice holds in L.*

Proof. L can be well ordered, and well ordering is equivalent to Choice. □

Remark. The well ordering can also be constructed explicitly, using Gödel operations. The proof is provided in the Appendix, Theorem 10.

6.2 GCH Theorem

Theorem 7. *GCH holds for L: if $V = L$ then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every α.*

Proof. What we want to show is that $|\mathcal{P}^L(\omega_\alpha)| = \aleph_{\alpha+1}$. First, let us prove that

$$|\mathcal{P}^L(\omega_\alpha)| \leq \aleph_{\alpha+1} \tag{13}$$

To do that, we would like to show that $\mathcal{P}^L(\omega_\alpha) \subset L_{\omega_{\alpha+1}}$.

If we can show this, then actually we are done. Because not only have we shown Equation (13), we have also shown that $P^L(\omega_\alpha) \in L$.

Remember that for a transitive model M, $\mathcal{P}^M(X) = \mathcal{P}(X) \cap M$ (because $x \subset Y$ is a Δ_0 formula). Thus $|\mathcal{P}^L(\omega_\alpha)| = |\mathcal{P}(\omega_\alpha)| > |\omega_\alpha| = \aleph_\alpha$. Further, this means $|\mathcal{P}^L(\omega_\alpha)| = |\mathcal{P}(\omega_\alpha)| = 2^{\aleph_\alpha}$.

Combine this result with Equation 13, we see that $|\mathcal{P}^L(\omega_\alpha)| = 2^{\aleph_\alpha} = \aleph_{\alpha+1}$, and we are done.

Claim. $\mathcal{P}^L(\omega_\alpha) \subset L_{\omega_{\alpha+1}}$

Proof. Take $x \subset \omega_\alpha$. Suppose we can get $\beta < \omega_{\alpha+1}$ such that $x \in L_\beta$, then we are done. We will proceed via Löwenheim-Skolem (Lemma 8), which is great for reducing collapsing sets down to smaller sets.

Take a limit ordinal λ such that $x \subset \omega_\alpha \subset L_\lambda$. Löwenheim-Skolem (Lemma 8) then gives an elementary submodel M such that $\omega_\alpha \subset M$ (hence $x \in M$), $M \prec L_\delta$, and $|M| \leq \aleph_\alpha$. [3]

Now take the Mostowski collapse of M, to get a transitive set N, such that $M \cong N \subset L_\lambda$.

Now consider the formula σ, which satisfies

$$M \models \sigma \iff M = L_\lambda \text{ for some limit ordinal } \lambda.$$

We have $L_\lambda \models \sigma$ trivially.

$\Rightarrow M \models \sigma$ because $M \prec L_\lambda$.

$\Rightarrow N \models \sigma$ as $N \cong M$.

$\Rightarrow N = L_\mu$ for some limit ordinal μ, because of how σ is defined.

$|N| = |M| \leq \aleph_\alpha$, but $M \supset \omega_\alpha$, so $|M| \geq \aleph_\alpha$. Thus $|N| = |M| = \aleph_\alpha$. N is transitive, so this means $\omega_\alpha \subset N$.

$\omega_\alpha \subset M$, so due to the uniqueness of the collapsing map in Mostowski theorem, we get that the collapsing map π is the identity on ω_α. Also remember that $x \subset \omega_\alpha$, so $\pi(x) = x$. But of course, $x = \pi(x) \in \pi(M) = N = L_\mu$.

$|N| = |\mu| = \aleph_\alpha < \aleph_{\alpha+1}$

Hence $x \in L_\mu$, where $|\mu| < \aleph_{\alpha+1}$, thus $\mathcal{P}^L(\omega_\alpha) \subset L_{\omega_{\alpha+1}}$ and we are done. □

□

References

[1] Thomas Jech, Set Theory, The Third Millennium Edition, Chapter 12 and 13.

[2] Kenneth Kunen, Set Theory, 1980, Chapter VI

[3] https://en.wikipedia.org/wiki/Constructible_universe

[3] **Warning!** When we are using Lowenstin Skolem, we are using the Axiom of choice, and we are not assuming choice in this article. However, this does not effect us, because we are working in L, and AC holds in L.

7 Appendix

7.1 Variation on the Reflection Principle

Theorem 8. *(Variation on Reflection Principle) Let ϕ be a formula. There are arbitrarily large limit ordinals α such that $\phi^L(x_1, ...x_n) \Leftrightarrow \phi^{L_\alpha}(x_1, ...x_n)$ for all $x_1, ...x_n \in L_\alpha$.*

Proof of Theorem 8. There are some very obvious choices of ϕ, for which this is trivially true: atomic formulae $x = y$ and $x \in y$ are fine, because we can just take and L_α which includes M_0. Suppose ϕ and ψ such that the theorem holds, then the theorem obviously holds for $\phi \wedge \psi, \neg\phi, \phi \vee \psi, \phi \Leftrightarrow \psi$ and $\phi \Rightarrow \psi$ too. This seems to point at using these building blocks to make ϕ, and we almost have all the components, except for $(\exists x)\phi$, and for that we use Lemma 1.

Once we have this lemma, let us take $\phi(x_1, ...x_n)$ and start to deconstruct it.

First of all, let us replace all the universal quantifiers $(\forall x)$ with $\neg\exists\neg$.

Next, look at all the subformulae of ϕ. Subformulae are defined inductively by the intuitive way: for example, $(\forall x)\phi$ has subformula ϕ, and $\phi \wedge \psi$ has subformula ϕ and ψ etc. We pick out all the smaller components of ϕ, right down to the atomic formulae. Suppose that its subformulae are $\phi_1, ...\phi_n$.

Now, we can show that each subformulae satisfy theorem and build up to ϕ, and the only problematic case would be the $(\exists x)$ case. By the lemma that we have just shown, there exist α such that for $j = 1, 2, ...n$ and $u... \in L_\alpha$,

$$(\exists x)\phi_j(u, ...x) \Rightarrow (\exists x \in L_\alpha)\phi_j(u...x) \tag{14}$$

Now we can easily show that L_α reflects $(\exists x)\phi_j$: Take $u_1...u_n \in L_\alpha$, then

$$\begin{aligned} L_\alpha \models \exists x\ \phi_j(u_1, ...u_m, x) &\Leftrightarrow (\exists x \in M)\ \phi_j^M(u_1, ...u_m, x) \\ &\Leftrightarrow (\exists x \in M)\phi_j(u_1, ...u_m, x) \\ &\Leftrightarrow \exists x\ \phi_j(u_1, ...u_m, x) \end{aligned}$$

We can take away the realisation there, because we have restricted u_i to M; the last equivalence is because of (1).

And now we just show that L_α is the set that reflects ϕ, that we are looking for. L_α reflects every atomic formulae, and if it reflects ϕ and ψ, then it reflects $\neg\phi, (\exists x)\phi, \phi\wedge\psi, \phi\vee\psi, \phi \Leftrightarrow \psi$ and $\phi \Rightarrow \psi$. So we can build ϕ using these simpler components, and L_α reflects it. $\square$

7.2 Gödel Normal Form Theorem

Theorem 9. *(Gödel's Normal Form Theorem)*

If $\phi(u_1, ...u_n)$ is a Δ_0 formula, then there is a composition G of $G_1, ...G_{10}$ such that for all $X_1, ..., X_n$,

$$G(X_1, ..., X_n) = \{(u_1, ..., u_n) : u_1 \in X_1, ...u_n \in X_n \text{ } and \text{ } \phi(u_1, ..., u_n)\}. \qquad (15)$$

Proof. If we have '$x = y$', we can replace this with '$(\forall u \in x) u \in y \wedge$ (for all$v \in y) v \in x$'; $x \in x$ can be replaced by $(\exists u \in x) u = x$.

Also, let us rename the bounded variables in $\phi(u_1, ...u_n)$ such that the variable with the highest index is quantified first.

Thus: all Δ_0 formulae can be written in one of the following forms [4]:

(i) The only logical symbols in ϕ are $\neg$, $\wedge$, and restricted $\exists$;

(ii) $=$ does not occur;

(iii) The only occurrence of $\in$ is $u_i \in u_j$, where $i \neq j$;

(iv) ϕ is of the form $\exists(u_{m+1} \in u_i)\psi(u_1, ..., u_{m+1})$ where $i \leq m$.

Now, we prove the theorem case by case.

Case I. $\phi(u_1, ...u_n)$ is an atomic formula $u_i \in u_j (i \neq j)$. We prove this case by induction on n.

<u>*Case Ia.*</u> $n = 2$. Here we have

$$\{(u_1, u_2) : u_1 \in X_1 \wedge u_2 \in X_2 \wedge u_1 \in u_2\} = \epsilon(X_1, X_2)$$

and

$$\{(u_1, u_2) : u_1 \in X_1 \wedge u_2 \in X_2 \wedge u_2 \in u_1\} = G_8(\epsilon(X_2, X_1)).$$

(These motivates the inclusion of ϵ and G_8 in the operations)

Now, we can use induction on n:

<u>*Case Ib.*</u> $n > 2, i, j \neq n$. By the inductive hypothesis, there is a G such that

$$\{u_1, ...u_{n-1} : u_1 \in X_1, ...u_{n-1} \in X_{n-1} \wedge u_i \in u_j\} = G(X_1, ..., X_{n-1}).$$

Then:

$$\{u_1, ...u_{n-1}, u_n : u_1 \in X_1, ...u_n \in X_n \wedge u_i \in u_j\} = G(X_1, ..., X_{n-1}) \times X_n.$$

<u>*Case Ic.*</u> $n > 2, i, j \neq n - 1$. The idea is very similar to the previous case, except we have to perform a swap.

Use the *Case Ib* to get G such that:

$$\{(u_1, ...u_{n-2}, u_n, u_{n-1}) : u_1 \in X_1, ...u_n \in X_n \text{ and } u_i \in u_j\} = G(X_1, ..., X_n)$$

But

$$(u_1, ...u_n) = ((u_1, ...u_{n-2}), u_n, u_{n-1})$$

[4]We allow dummy variables, so $\phi(x_1, ...x_n) = (x_1 \in x_2)$ and $\phi(x_1, ...x_{n+1}) = (x_1 \in x_2)$ are different formulae.

We get

$$\{u_1, ..., u_n : u_1 \in X_1, ...u_n \in X_n \wedge u_i \in u_j\} = G_9(G(X_1, ..., X_{n-1})).$$

<u>*Case Id.*</u> $i = n-1, j = n$. By *Case Ia,* we have
$\{(u_{n-1}, u_n) : (u_{n-1} \in X_{n-1} \wedge u_n \in X_n \wedge u_{n-1} \in u_n)\} = \epsilon(X_{n-1}, X_n)$
So

$$\begin{aligned}\{(u_{n-1}, u_n), (u_1, ...u_{n-2}) : u_1 \in X_1, ...u_n \in X_n \wedge u_{n-1} \in u_n)\}\\ = \epsilon(X_{n-1}, X_n) \times (X_1 \times ...X_{n-2}) = G(X_1, ...X_n)\end{aligned}$$

Note that

$$((u_{n-1}, u_n), (u_1, ...u_{n-2}) = (u_{n-1}, u_n, (u_1, ...u_{n-2}))$$

and

$$(u_1, ...u_n) = ((u_1, ...u_{n-2}), u_{n-1}, u_n)$$

So we swap using G_{10}:

$$\{(u_1, ...u_n) : u_1 \in X_1, ...u_n \in X_n \wedge u_{n-1} \in u_n)\} = G_{10}(G(X_1, ..., X_n))$$

Case Ie. $i = n, j = n-1$: similar to the previous case.

Case II. $\phi(u_1, ..., u_n)$ is a negation, $\neg\psi(u_1, ...u_n)$. By the induction hypothesis, there is a G such that:

$$\{u_1, ..., u_n : u_1 \in X_1, ...u_n \in X_n \text{ and } \psi(u_1, ...u_n)\} = G(X_1, ...X_n).$$

The set we want is easily achieved via G_4:

$$\begin{aligned}\{u_1, ..., u_n : u_1 \in X_1, ...u_n \in X_n \text{ and } \neg\psi(u_1, ...u_n)\}\\ = X_1 \times ... \times X_n - G(X_1, ...X_n).\end{aligned}$$

Case III.$\phi = \psi_1 \wedge \psi_2$. This case is very easy to handle, using the induction hypothesis, and G_5 intersection.

Case IV. $\phi(u_1, ...u_n)$ is the formula $(\exists u_{n+1} \in u_i)\psi(u_1, ...u_{n+1})$.

Let $\chi(u_1, ...u_{n+1})$ be the formula $\psi(u_1, ...u_{n+1}) \wedge u_{n+1} \in u_i$. By induction hypothesis (χ is less complex than ϕ, because there is one less quantifier), there is a G such that

$$\begin{aligned}\{(u_1, ...u_{n+1}) : u_1 \in X_1, ..., u_{n+1} \in X_{n+1} \text{ and } \chi(u_1, ...u_{n+1})\}\\ = G(X_1, ...X_{n+1})\end{aligned}$$

For all $X_1, ...X_{n+1}$. We claim that

$$\begin{aligned}\{(u_1, ...u_n) : u_1 \in X_1, ...u_n \in X_n \text{ and } \phi(u_1, ...u_n)\}\\ = (X_1 \times ... \times X_n) \cap \mathrm{dom}(G(X_1, ...X_n, \bigcup X_i))\end{aligned} \tag{16}$$

Let us denote $u = (u_1, ..., u_n)$ and $X = X_1, ...X_n$. For all $u \in X$, we have

$$\begin{aligned}\phi(u) &\Leftrightarrow (\exists v \in u_i)\psi(u_i, v) \\ &\Leftrightarrow \exists v(v \in u_i \wedge \psi(u, v) \wedge v \in \bigcup X_i) \\ &\Leftrightarrow u \in \mathrm{dom}\{(u, v) \in X \times \bigcup X_i : \chi(u, v)\}\end{aligned}$$

And (16) follows, and this completes the proof. □

7.3 An Explicit Well Ordering of L

Theorem 10 (Gödel)**.** *There exists a well-ordering of L.*

Proof. $L = \bigcup_{\alpha \in Ord} L_\alpha$, it is a nested hierarchy, so it is natural to try to order each L_α first, then make the orderings compatible.

So we want to construct, inductively, a well ordering $<_\alpha$ for each L_α, and we do it in a way such that if $\alpha < \beta$, then $<_\beta$ is an *end-extension* of $<_\alpha$, i.e.:

(i) if $x <_\alpha y$, then $x <_\beta y$

(ii) If $x \in L_\alpha$, $y \in L_\beta - L_\alpha$, then $x <_\beta y$

So: if $\alpha < \beta$, then everything in L_α are 'smaller'.

Also: if $x \in y \in L_\alpha$, then: due o transitivity, $\exists \beta < \alpha$ such that $x \in L_\beta$, thus $x <_\alpha y$.

Now, the rest is just making our intuition concrete. For uninterrupted reading feel free to skip the rest of the proof.

Step 1: Limit ordinal

Suppose the λ is a limit ordinal, and for all $\alpha < \lambda$ we have $<_\alpha$ that satisfies our conditions, then simply let $<_\lambda = \bigcup_{\alpha <_\lambda}$.

Step 2: Define $<_{\alpha+1}$

We recall the definition of $L_{\alpha+1}$, based on Gödel operation:

$$L_{\alpha+1} = \mathrm{def}(L_\alpha) = \mathcal{P}(L_\alpha) \cap \mathrm{cl}(L_\alpha \cup \{L_\alpha\})$$

We can further break down the definition for $\mathrm{cl}(M)$ by looking at the construction of $\mathrm{cl}(M)$ step by step. We define W_n^α as follows:

(i) $W_0^\alpha = L_\alpha \cup \{L_\alpha\}$

(ii) $W_{n+1}^\alpha = \{G_i(X, Y) : X, Y \in W_n^\alpha, i = 1, ...10\}$

Then we note that $L_{\alpha+1} = \mathcal{P}(L_\alpha) \cap \bigcup_{n=0}^\infty W_n^\alpha$.

Why do we do this?

the closure of $L_\alpha \cup \{L_\alpha\}$ imposes structure on L_α: when looking at closure, we are constructing sets step by step using G_i, giving rise to a natural order.

Following this intuition, let us order L_α:

First we take elements of L_α, then L_α itself, then the remaining elements of W_1^α, then the remaining elements of W_2^α etc.

Now let us order the elements in W_{n+1}^α. We notice that very important fact that every $x \in W_{n+1}^\alpha$ is equal to $G_i(u,v)$ for some u, v and $i = 1, ..., 10$. This is again great, because $G_i's$ are numbered, giving another natural order.
But when even that fails, we compare the least u's such that $x = G_i(u,v)$. When that fails again, we compare the last v such that $x = G_i(u,v)$. At this point, this must work, and this gives the desired ordering.

Here comes the tedious definition, which described the intuition above:

(i) $<_{\alpha+1}^0$ is the well-ordering of $L_\alpha \cup \{L_\alpha\}$ that extends $<_\alpha$ and such that L_α is the last element.

(ii) $<_{\alpha+1}^n$ is the following well-ordering of W_{n+1}^α:
$x <_{\alpha+1}^{n+1} y$ iff either:

- $x <_{\alpha+1}^n y$ OR:
- $x \in W_n^\alpha$ and $y \notin W_n^\alpha$ OR:
- $x \notin W_n^\alpha$ and $y \notin W_n^\alpha$ AND:
  (a) the least i such that $\exists u, v \in W_n^\alpha (x = G_i(u,v)) <$
  the least j such that $\exists s, t \in W_n^\alpha (y = G_i(s,t))$ OR:
  (b) the least $i =$ the least j and
  [the $<_{\alpha+1}^n$-least $u \in W_n^\alpha$ such that $\exists v \in W_n^\alpha (x = G_i(u,v))$] $<_\alpha^n$
  [the $<_{\alpha+1}^n$-least $s \in W_n^\alpha$ such that $\exists t \in W_n^\alpha (y = G_i(s,t))$], OR:
  (c) the least $i =$ least j and the least $u =$ least s and
  [the $<_{\alpha+1}^n$-least $v \in W_n^\alpha$ such that $x = G_i(u,v)$] $<_{\alpha+1}^n$
  [the $<_{\alpha+1}^n$-least $t \in W_n^\alpha$ such that $x = G_i(u,t)$]

Now we are tempted to take $\bigcup_n <_{\alpha+1}^n$, but in that case, what we have is a ordering of $\bigcup_{n=0}^\infty W_n^\alpha$. We want an ordering of $L_{\alpha+1} = \mathcal{P}(L_\alpha) \cap \bigcup_{n=0}^\infty W_n^\alpha$.
Also remember, that $<$ is essentially a set of ordered pairs, comparing each subset of L_α.
Thus we will define the following to get an ordering of $L_{\alpha+1}$:

$$<_{\alpha+1} = \bigcup_{n=0}^{\infty} <_{\alpha+1}^n \cap (\mathcal{P}(L_\alpha) \times \mathcal{P}(L_\alpha))$$

It is clear that $<_{\alpha+1}$ is an end-extension of $<_\alpha$ and is a well-ordering of $L_{\alpha+1}$.

Now having defined $<_\alpha$ for all α, we let $x <_L y \Leftrightarrow \exists \alpha\, x <_\alpha y$. This relation $<_L$ is a well-ordering of L. $\square$