Mcintyre's Theorem

Tianyiwa Xie

1 Unfinished from Last Time

Lemma 1.1.

1. *G has a unique generic type iff G is connected.*
2. *$deg_M(G) = [G : G^0]$*

Proof. The first part we have proven the last time. Now we prove the second part.

Remember $\deg_M(\phi) = |\{p \in S_n(G) : MR(p) = \mathrm{MR}(\phi) \wedge \phi \in p\}|$ [1, Lemma 6.2.16].

So since G^0 has a unique maximal rank, $\deg_M(G^0) = 1$. But because G is a union of $[G : G^0]$ disjoint translates of G^0, $\deg_M(G) = [G : G^0]$. □

Lemma 1.2. *If $g \in G$, there are $a, b \in \mathbb{G}$ generic over G such that $g = ab$.*

Proof. Let $a \in \mathbb{G}$ be generic over G. Because $x \mapsto x^{-1}g$ is a definable bijection, $b = a^{-1}g$ is also generic over G and $g = ab$. □

Remark. Not only can we always find elements a, b, any generic element will give one such pair.

Definition 1. We say that a definable $A \subseteq G$ is generic if $\mathrm{MR}(A) = \mathrm{MR}(G)$.

Let me mention a perhaps obvious fact:

Fact 1. *If G is connected and $A \subset G$ is a generic set defined by $\phi(v)$, and p is the unique type with maximal rank, then $\phi \in p$.*

Proof. Suppose not, then $\neg\phi \in p$ and $\mathrm{MR}(\neg\phi) = \mathrm{MR}(G)$. But $\neg\phi(G) = A^c$, so $\mathrm{MR}(A^c) = \mathrm{MR}(A) = \mathrm{MR}(G)$, meaning the Morley degree of $G \geq 2$. Contradiction since connectedness means $\deg_M(G) = 1$. □

Lemma 1.3. *Suppose that G is connected and $A \subseteq G$ is a definable generic subset. Then $G = A \cdot A = \{ab : a, b \in A\}$.*

Proof. Let ϕ be a formula defining A. From the previous lemma, we know that for all $g \in G$, can find $a, b \in \mathbb{G}$ generic such that $g = ab$.

But because there is only one unique generic type p and $\mathrm{MR}(\phi) = \mathrm{MR}(G)$, $\phi \in p = \mathrm{tp}(a/G) = \mathrm{tp}(b/G)$. So $\mathbb{G} \models \phi(a) \wedge \phi(b)$.

So $\mathbb{G} \models \exists x \exists y(\phi(x) \wedge \phi(y) \wedge xy = g)$. Because $G \prec \mathbb{G}$, there are $a', b' \in A$ such that $g = a'b'$. □

We will now show that finitely many translates of a generic set cover the group. But to do that we need a result on Stone spaces and a results on unique nonforking extensions:

Definition 2 (Stone Topology)**.** For ϕ a $\mathcal{L}_A$ formula with free variables from $v_1, ..., v_n$, let $[\phi] = \{p \in S_n^{\mathcal{M}}(A) : \phi \in p\}$.

Let $\mathcal{M}$ be a $\mathcal{L}$-structure and $A \subset M$. The Stone topology on $S_n^{\mathcal{M}}(A)$ is the topology generated by taking the sets $[\phi]$ as open sets. In fact these sets are also closed.

Fact 2. *$S_n^M(A)$ is compact.*

Lemma 1.4.

1. *Suppose that $p \in S_n(A)$ and $A \subset B$, and $deg_M(p) = 1$. Then p has a unique nonforking extension in $S_n(B)$ [1, Theorem 6.3.2(iii)].*
2. *Further, this nonforking extension is given by $p_B = \{\Phi(\overline{x}, \overline{b}) : \mathbb{M} \models d_p\Phi(\overline{b}), \overline{b} \in B\}$, where $d_p\Phi(\overline{y})$ is the formula defining whether $\Phi(\overline{x}, \overline{y}) \in p$ [1, Proposition 6.3.7].*

Lemma 1.5. *Let $A \subset G$ be a definable generic subset of G, then there are $a_1, a_2, \dots, a_n \in G$ such that $G = a_1A \cup a_2A \cup \cdots \cup a_nA$.*

Proof. Because finitely many translates of G^0 cover G, we may, without loss of generality, assume that G is connected. Then we know that G has a unique type with maximal MR, and also $\deg_M(G) = 1$.

Let $\phi(x)$ be the $\mathcal{L}_G$-formula defining A. Let $p \in S_1(G)$ be the unique generic type. From Fact 1, $\phi \in p$.

Claim 1. *For any $q \in S_1(G)$, there is $g \in G$ such that $\phi(gv) \in q$.*

Remark. The idea behind this claim is to get a correspondence between elements in G and the types in $S_1(G)$. Then we will make use of the compactness of the Stone space to finish the proof.

Proof. By the definability of types, for all $\mathcal{L}_G$ formula $\Phi(x, \overline{y})$, there is a $\mathcal{L}_G$ formula $d_q\Phi(\overline{y})$ such that $\Phi(x, \overline{a}) \in q \iff \mathbb{G} \models d_q\Phi(\overline{a})$.

Let a, b be independent realizations of p, q. This means that $q = \operatorname{tp}(b/G)$ and $\operatorname{tp}(b/G \cup \{a\})$ is nonforking. In particular, since $\deg_M(G) = 1$, this nonforking extension is unique, and $\operatorname{tp}(b/G \cup \{a\}) = \{\Phi(x, a) : \mathbb{G} \models d_q\Phi(a)\}$ [1, Theorem 6.3.8].

Because a is generic, ab is generic, so $\operatorname{tp}(ab/G)$ is generic and $\phi \in \operatorname{tp}(ab/G)$. Therefore $\mathbb{G} \models \phi(ab)$.

Let $\psi(v, w)$ be the formula $\phi(w \cdot v)$ and $d_q\psi$ be such that $\psi(v, g) \in q$ iff $\mathbb{G} \models d_q\psi(g)$.

Because $\mathbb{G} \models \phi(ab) = \psi(b, a)$, $\psi(b, a) \in \operatorname{tp}(b/G \cup \{a\})$ and equivalently $\mathbb{G} \models d_q\psi(a)$.

Thus $\mathbb{G} \models \exists w d_q\psi(w)$. Because $G \prec \mathbb{G}$, there exists $g \in G$ such that $\phi(gv) \in q$. □

For each $g \in G$, we let $O_g = \{q \in S_1(G) : \phi(gv) \in q\}$. This is an open subset of $S_1(G)$ and by the claim, $S_1(G) = \cup_{g \in G} O_g$.

By compactness of Stone spaces, we get $a_1, \dots, a_n \in G$ such that $S_1(G) = O_{a_1} \cup \dots \cup O_{a_n}$.

In particular, if $g \in G$, and q' is the unique type containing the formula $v = g$, there is an i such that $q \in O_{a_i}$. But then $\phi(a_i g)$ and $g \in a_i^{-1}A$.

Thus $G = a_1^{-1}A \cup \dots \cup a_n^{-1}A$. □

2 Preliminaries from Model Theory

In this section, G means a ω-stable subgroup and $\mathbb{G}$ is a monster model such that $G \prec \mathbb{G}$; $\mathcal{M}$ means a ω-stable model, with monster model $\mathbb{M}$ and $M \prec \mathbb{M}$.

Lemma 2.1. *Suppose G is a ω-stable group and $H \leq G$ is a definable subgroup, then for any coset aH, $MR(aH) = MR(H)$.*

Lemma 2.2. *Suppose G is an ω-stable group and $\sigma : G \to G$ is a definable group automorphism. Then σ fixes G^0 set wise.*

Proof. Take $\sigma(G^0) \cap G^0$. This is a definable finite index subgroup of G. But G^0 is the smallest, so $G^0 \subset \sigma(G^0)$.

However, $[G : \sigma(G^0)] = [G : G^0]$. This is because σ is a bijection. So $G^0 = \sigma(G^0)$. □

Lemma 2.3. *Suppose $\mathcal{M}$ is ω-stable, and $X \subseteq \mathbb{M}^n, Y \subseteq \mathbb{M}^n$ are definable, and $f : X \to Y$ is a definable finite-to-one function from X to Y. Then $MR(X) = MR(Y)$.*

This final Lemma was mentioned in Lecture number 7. Alternatively, see [1, Theorem 6.2.18].

3 Preliminaries from Galois Theory

The proof of Mcintyre's theorem relies on results from Galois theory, which we will state here:

Lemma 3.1. *Suppose L/K is a Galois extension with degree n, and q is a prime dividing n, then there exists a field $K \subseteq F \subset L$ such that L/F is Galois of degree q.*

Additionally the Galois group L/F is cyclic.

Proof. This is a result of the fundamental theorem of Galois theory and Cauchy theorem. □

Lemma 3.2. *Every finite extension of a perfect field is separable. Thus the algebraic closure of a perfect field is normal by definition and separable, so it is Galois.*

Theorem 3.1.

(a) *Suppose that L/K is a cyclic Galois extension of degree n, where n is coprime to char(K) and K contains all nth roots of unity. Then minimal polynomial L/K is $X^n - a$ for some $a \in K$.*

(b) *Suppose that K has characteristic $p > 0$ and L/K is a Galois extension of degree p. Then the minimal polynomial of L/K is $X^p - X - a$ for some $a \in K$.*

4 ω-stable Fields

In this section, we assume that $(K, +, \cdot, \dots)$ is a ω-stable, infinite field. We will prove a series of results about ω-stable fields, culminating in Mcintyre's Theorem.

Lemma 4.1. *K is connected.*

Proof. Suppose that K^0 is a connected component of the additive group. For $a \in K \setminus \{0\}$, $x \mapsto ax$ is a definable group automorphism.

Therefore, by Lemma 2.2, K^0 is closed under this map. Therefore K^0 is an ideal of K. But K is a field which does not have any proper ideals, so $K^0 = K$. $\square$

Corollary 4.1.1. *The multiplicative group $(K^\times, \cdot, \dots)$ is connected.*

Proof. By Lemma 1.1, $K^\times$ has a unique generic type iff $K^\times$ is connected.

Lemma 4.1 shows K has a unique maximal type and Morley degree 1. Now suppose that $K^\times$ have two different complete types p and q with maximal rank, then there exists a formula $\phi \in p$ and $\neg\phi \in q$. But then ϕ and $\neg\phi$ both have maximal rank, giving Morley degree of $K \geq 2$, contradiction. $\square$

Now, we prove a lemma that follows easily from Lemma 2.3.

Lemma 4.2. *Suppose $(G, \cdot, \dots)$ is a ω-stable group which is connected, and suppose $f : G \to G$ is a definable, finite to one group homomorphism. Then $f(G) = G$.*

Proof. $f(G) \leq G$ is a definable subgroup, so if $f(G)$ is also a finite index subgroup, we are done due to connectedness of G. But unfortunately we do not know this (e.g. consider $(\mathbb{Q}^\times)^m$ as a subgroup of $\mathbb{Q}^\times$).

But by Lemma 2.3, $\mathrm{MR}(f(G)) = \mathrm{MR}(G)$. Therefore $[G : f(G)]$ must be finite and $f(G) = G$. $\square$

Corollary 4.2.1. *For every natural number n, $K^n = K$, i.e. every element in K has an nth root. In particular, K is perfect.*

Proof. $x \mapsto x^n$ is a multiplicative homomorphism, which is finite to one, so by Lemma 4.2, $(K^\times)^n = (K^\times)$, and $K^n = K$. $\square$

Corollary 4.2.2. *Suppose that K has characteristic $p > 0$, then the map $x \mapsto x^p - x$ is surjective.*

Proof. $f : G \to G$ given by $f(x) = x^p - x$ is an finite to one, additive homomorphism. So by Lemma 4.2, $f(G) = G$ and f is surjective. $\square$

Lemma 4.3. *Suppose that L/K is a finite extension, then L is also ω-stable.*

Proof. L is a finite vector space over K, say with dimension n and basis $\{e_1, \dots, e_n\}$. So we can then interpret L in K:

Define addition as $(a_i)_{1\leq i\leq n} + (b_i)_{1\leq i\leq n} = (a_i + b_i)_{i\leq n}$, and multiplication according to the behavior of the coefficients in L. These are definable operations.

More specifically, we can define addition and multiplication on K^n such that the map $f : L \to K^n$, where $\sum_{i=1}^n a_i e_i \mapsto (a_1, \dots, a_n)$ preserves addition and multiplication.

Therefore L is interpretable in K. Now if we take any $A \subseteq K^n$ with $|A| = \omega$, all types in $S_m^{(K^n)}(A)$ correspond to a type in K over a countable set. Therefore $|S_m^{(K^n)}(A)| = \omega$, and L is ω-stable. $\square$

Theorem 4.1 (Mcintyre)**.** *$(K, +, \cdot, \dots)$ is algebraically closed.*

Proof. Lemma 4.1 and Corollary 4.1.1 tells us that K and $K^\times$ are connected. Corollary 4.2.1 tells us K is perfect.

It is enough to show that K has no proper Galois extensions: we have mentioned that every finite extension of this field is Galois, so if there are no proper Galois extensions, we must have $\overline{K} = K$.

Claim 2. *Suppose that K is an infinite ω-stable field, containing all mth roots of unity for all $m \leq n$. Then K has no proper Galois extensions of degree n.*

Proof. Let n be the least such that there is a ω-stable field containing all mth roots of unity for $m \leq n$ and K has a proper Galois extension L of degree n.

Let q be a prime number dividing n. By Galois theory, there is a $K \subseteq F \subset L$ such that L/F is Galois of degree q.

The field F is a finite algebraic extension of K, so it is also ω-stable by Lemma 4.3. Thus, by the minimality of n, $F = K$, and $n = q$, and consequently L/K is cyclic.

Now we split into two cases.

Case 1: $\text{char}(K) = 0$ or $\text{char}(K) = p \neq q$.

By Theorem 3.1(a), the minimal polynomial of L/K is $X^q - a$ for some $a \in K$. But every element in K has a qth root, so $X^q - a$ is reducible, a contradiction.

Case 2: K has characteristic $p = q$. Then by Theorem 3.1(b), the minimal polynomial of L/K is $X^p - X - a$ for some $a \in K$. But from Corollary 4.2.2, we know that this map is surjective, so reducible, contradiction. □

Claim 3. *If K is an infinite ω-stable field, then K contains all roots of unity.*

Proof. Let n be the least such that K does not contain all nth roots of unity. Let ξ be a primitive root of unity.

Then $K(\xi)$ is a Galois extension of K of degree at most $n - 1$. This contradicts Claim 1. □

So K contains all roots of unity, and Claim 1 proves that K has no proper Galois extensions. Thus we are done. □

5 ω-stable Groups

We have shown in this section that ω stability in infinite fields imply algebraically closure. The converse is not true.

For an infinite group with no proper definable infinite subgroups, ω-stable implies G is Abelian. This result implies that if G is an infinite ω-stable group, then there is an infinite definable Abelian $H \leq G$.

References

[1] D. Marker. Model theory: An introduction. 217, 01 2002.